Liverpool John Moores University
SCHOOL OF COMPUTING AND MATHEMATICAL SCIENCES

Referral Examinations, August 1997

CMSMA200

Numerical Algorithms and their Analysis

Duration 2 Hours

Instructions to candidates

Do not open this question paper until you have been told to do so by the invigilator.
A figure in [] denotes the number of marks available for that question or part of question.

There are 5 questions.
Answer 4 questions.
Questions carry 25 marks each.
The total number of marks available is 100

You have the use of the Software DERIVE or DERIVE for Windows. You may not use a printer to print out any mathematical expressions or graphs.

Any results or graphs that DERIVE produces, on screen, that are relevant to your examination work should be written or sketched in your answer book.

You may also use **any** calculator and its memory need not be erased before the examination.

1.(a) (i) With the aid of a diagram, describe the Mid point rule for estimating $\int_a^b f(x)\,dx$. Also, show that the composite Midpoint rule with n strips can be written as

$$M_n = \sum_{r=1}^{n} hf\left(a + \frac{(2r-1)h}{2}\right)$$

where $h = \dfrac{(b-a)}{n}$. [5]

Below is a table of values for the function $\sqrt{|\cos x|}$ between x=0 and x=π, in steps of $\dfrac{p}{8}$.

x	0	$\frac{p}{8}$	$\frac{p}{4}$	$\frac{3p}{8}$	$\frac{p}{2}$	$\frac{5p}{8}$	$\frac{3p}{4}$	$\frac{7p}{8}$	p
$f(x)$	1	0.961186	0.840896	0.618614	0	0.618614	0.840896	0.961186	1

(ii) Using the composite Midpoint rule with four strips, M_4, and the above table of values, show (using paper, pencil and a calculator **only**) that

$$\int_0^p \sqrt{|\cos x|}\,dx \approx 2.4815.$$ [7]

(iii) Use the composite Mid point rule with 8 strips, M_8, to find a further estimate of $\int_0^p \sqrt{|\cos x|}\,dx$. (You may use any DERIVE functions that you have developed throughout the course to calculate M_8) [4]

(iv) Given that the order of error of the midpoint rule is h^2, show by Richardson's extrapolation that a further estimate for this integral can be found from the expression

$$\frac{4M_8 - M_4}{3}.$$

Hence find this further approximation. [9]

2. (a) Write down, or derive, the **central** difference formula for $\frac{dy}{dx}$.

Show that the central difference formula can be derived from the forward and backward difference formulas for $\frac{dy}{dx}$. [3]

(b) Use the central difference formula to approximate the derivative of

$$e^x \ln x$$

at x=0, with a step size of h=0.1. [3]

(c) Given that the error term of the central difference formula, with step size h, applied to a function $f(x)$ is: $\frac{h^2}{6}M$, where M is the maximum value of $|f'''(x)|$, in the region $[x-h, x+h]$. Find an upper bound for the error of the answer in (b). [4]

(d) Show, in operator algebra, that $f(x+h) = e^{hD} f(x)$ where the operator D is the derivative operator, $Df(x) = \frac{d}{dx} f(x)$.

Also, show that $\Delta f(x) = (e^{hD} - 1) f(x)$, where

$\Delta f(x) = f(x+h) - f(x)$,

defines the forward difference operator Δ. Hence derive the formula

$$D = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right)$$ [5]

(e) Use the quadratic approximation to D, to show that derivative of $f(x)$, can be approximated by the formula

$$\frac{4f(x+h) - 3f(x) - f(x+2h)}{2h}$$ [4]

(f) Find the Taylor expansion of the above quadratic approximation up to terms in h^3. Hence, write down the order of error of the above approximation to the derivative of $f(x)$. [5]

(g) Stating a reason, which would you expect to be the more accurate, the central difference formula or the quadratic formula derived in (e) above?

3. (a) Using a 3 x 3 set of linear equations as an example, describe the Gauss Seidel method for solving sets of linear equations, highlighting the main difference between it and the Jacobi method.

Explain the condition of diagonal dominance with respect to the convergence of the Gauss Seidel method, in the case where there are:

(i) Two equations with two unknowns,
(ii) n equations in n unknowns.

[8]

(b) The Gauss Seidel method is applied to two linear simultaneous equations, and the iterative scheme

results. By recasting this iterative scheme into two simultaneous equations, show that the iterative scheme above will converge.

$$x^{(n+1)} = \frac{1}{2}\left(2 - y^{n}\right),\ y^{(n+1)} = \frac{1}{2}\left(2 + x^{(n+1)}\right)$$

The same simultaneous equations are rewritten into a new Gauss Seidel iterative scheme,

$$x^{(n+1)} = 2\left(y^{(n)} - 1\right),\ y^{(n+1)} = 2\left(1 - x^{(n+1)}\right)$$

Show that this *may not* converge. [6]

(c) Re-arrange the following linear equations into a suitable form to solve using the Gauss Seidel method.

$$x + y + 3z = 12$$
$$2x - y + z = 3$$
$$x - 2y - z = -6$$

With initial values of $x^{(0)} = 0$, $y^{(0)} = 0$ and $z^{(0)} = 0$, use pencil, paper and calculator only, to calculate the first iterates $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$.

[9]

Using DERIVE, or otherwise, find the solution to the above linear equations using the Gauss Seidel method. [2]

4.(a) (i) Show that the equation $e^x \ln x - \sin x = 0$ has at least one solution in the interval [1,2]. (*A graph alone will not gain full marks*) [3]

(ii) Describe, with the aid of a sketch, the Newton Raphson iterative method for solving equations. Describe a situation in which the Newton Raphson method will fail to produce a solution for a particular equation. [5]

(iii) Establish the Newton Raphson iterative formula for the equation $e^x \ln x - \sin x = 0$.
Using a starting value of $x_0 = 1.5$, find each of the Newton Raphson iterates x_1, x_2, x_3 and x_4, to 5 decimal places.

Write down the solution to the equation $e^x \ln x - \sin x = 0$, correct to 4 decimal places. Confirm that this solution is correct to 4 decimal places. [7]

(b) The equation $2x - e^{\sin x} = 0$ is to be solved using the Fixed Point Iteration method. The rearrangement (iterative scheme) chosen is $x_{n+1} = \frac{1}{2}e^{\sin x_n}$ and the starting value is $x_0 = 1$.

(i) Given that the solution to this equation is very close to $x = 1.32$, show that the above iterative scheme should converge to a solution. [3]

(ii) To 6 decimal places the first few iterates of the above scheme are

n	$x_{n+1} = \frac{1}{2}e^{\sin x_n}$
0	1.000000
1	1.159888
2	1.250554
3	1.291780
4	1.307582
5	1.313127
6	1.315003

Show that $\frac{e_4}{e_3} \approx 0.384$, where e_n is the error in approximating the solution by x_n. By also determining $\frac{e_5}{e_4}$ and $\frac{e_6}{e_5}$, obtain an estimate value for
$C = \lim_{n \to \infty} \frac{e_{n+1}}{e_n}$.

State the order of convergence of this series of iterates. [7]

5. (a) Describe fully, with reference to a sketch, the modified Euler method for solving first order differential equations, with initial conditions $y(x_0) = y_0$.

You are given the differential equation $\frac{dy}{dx} = 2(x^2 + y)$ with initial conditions $y(1) = 0$.

Use **paper, pencil and calculator** methods to show that the Euler method will yield an approximate solution to the above differential equation of $y(1.2) \approx 0.482$, using a step size h=0.1.

Find a further estimate (using DERIVE if you wish), for $y(1.2)$, using the Euler method with a step size h=0.05. [8]

(b) The order of convergence of the Euler method is h^2.

Use the order of convergence of the Euler method to extrapolate a more accurate solution of $y(1.2)$, using your answers above. [6]

(c) Use the Taylor series method order 4 to show that the solution to the differential equation $\frac{dy}{dx} = 2(x^2 + y)$ can be approximated by

$$y(x+h) \approx y(x) + 2h(x^2 + y) + 2h^2(x^2 + x + y) + \frac{4h^3}{3!}(2x^2 + 2x + 2y + 1) + \frac{8h^4}{4!}\left(2x^2 + 2x + 2y + 1\right)$$

where h is the step size.

Use the initial conditions $y(1) = 0$, a step size of 0.2 and the above expansion, to estimate $y(1.2)$. [11]