1. (a) Using a 3 x 3 set of linear equations as an example, describe the Jacobi method for solving sets of linear equations.

Explain the condition of diagonal dominance with respect to the convergence of the Jacobi method. [5]

(b) The Jacobi method is applied to two linear simultaneous equations, and the iterative scheme

$$x^{(n+1)} = \frac{1}{3}\left(3 - y^{n}\right),\ y^{(n+1)} = \frac{1}{4}\left(2 - 3x^{(n)}\right)$$

results. By recasting this iterative scheme into two simultaneous equations, show that the iterative scheme above will converge.

The same simultaneous equations are rewritten into a new Jacobi iterative scheme,

$$x^{(n+1)} = \frac{1}{3}\left(2 - 4y^{(n)}\right),\ y^{(n+1)} = 3\left(x^{(n+1)} - 1\right)$$

Show that this *may not* converge. [8]

(c) Re-arrange the following linear equations into a suitable form to solve using the Jacobi method.

$$\begin{aligned} x + 8y + z &= 12 \\ 6x - y + 3z &= 4 \\ x - 0.2y - z &= -6 \end{aligned}$$

With initial values of $x^{(0)} = 0$, $y^{(0)} = 0$ and $z^{(0)} = 0$, use pencil, paper and calculator only, to calculate the first iterates $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$.

Using DERIVE, or otherwise, find the solution to the above linear equations using the Jacobi method. [12]

2(a) (i) With the aid of a diagram, describe the Mid point rule for estimating $\int_a^b f(x)\,dx$. Also, show that the composite Midpoint rule with n strips can be written as

$$M_n = \sum_{r=1}^{n} hf\left(a + \frac{(2r-1)h}{2}\right)$$

where $h = \frac{(b-a)}{n}$. [5]

Below is a table of values for the function $\sqrt{\sin x}$ between x=0 and x=π, in steps of $\frac{p}{8}$.

x	0	$\frac{p}{8}$	$\frac{p}{4}$	$\frac{3p}{8}$	$\frac{p}{2}$	$\frac{5p}{8}$	$\frac{p}{2}$	$\frac{7p}{8}$	p
$f(x)$	0	0.619	0.841	0.961	1	0.961	0.841	0.619	0

(ii) Using the composite Midpoint rule with 4 strips, M_4, and the above table of values, show (using paper, pencil and a calculator **only**) that

$$\int_0^p \sqrt{\sin x}\; dx \approx 2.4819.$$ [7]

(iii) Use the composite Mid point rule with 8 strips, M_8, to find a further estimate of $\int_0^p \sqrt{\sin x}\; dx$. (You may use any DERIVE functions that you have developed throughout the course to calculate M_8) [4]

(iv) Given that the order of error of the midpoint rule is h^2, show by Richardson's extrapolation that a further estimate for this integral can be found from the expression

$$\frac{4M_8 - M_4}{3}.$$

Hence find this further approximation. [9]

3. (a) Use the power method to find the dominant eigenvalue and its associated eigenvector of the matrix

$$A = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

using the initial vector $\underline{v} = (1,1,1)^T$.

By finding the inverse of *A*, apply the power method using the initial vector $\underline{v} = (1,1,1)^T$,to find an approximation of the eigenvalue of smallest magnitude. [10]

(b) Recast the second order differential equation given below into two first order differential equations

$$y'' - 2x^2 y' + xy = 3\sin x \quad \text{for } y(0) = 0 \text{ and } y(1) = 0.$$

Using the RK function found in ODE_APPR.MTH file and the shooting method, numerically solve the above boundary value problem, using a step size of *h*=0.1.

Provide a sketch of the solution of $y(x)$ for $0 \le x \le 1$. [15]

4. (a) Describe fully, with reference to a sketch, the Euler's method for solving first order differential equations, with initial conditions $y(x_0) = y_0$.

You are given the differential equation $\frac{dy}{dx} = 2(x^2 + y)$ with initial conditions $y(1) = 0$.

Use **paper, pencil and calculator** methods to show that the Euler method will yield a solution to above differential equation of $y(1.2) \approx 0.482$, using a step size h=0.1.

Find a further estimate (using DERIVE if you wish), for $y(1.2)$, using the modified Euler method with a step size h=0.05. [8]

(b) The order of convergence of the Euler method is h.

Use the order of convergence of the Euler method to extrapolate a more accurate solution of $y(1.2)$ using your answers above. [6]

(c) Use the Taylor series method order 4 to show that the solution to the differential equation $\frac{dy}{dx} = 2(x^2 + y)$ can be approximated by

$$y(x+h) \approx y(x) + 2h(x^2 + y) + 2h(x^2 + x + y) + \frac{4h^3}{3!}(2x^2 + 2x + 2y + 1) + \frac{8h^4}{4!}\left(2x^2 + 2x + 2y + 1\right)$$

where h is the step size.

Use the initial conditions $y(1) = 0$, a step size of 0.2 and the above expansion to estimate $y(1.2)$. [11]

5.(a) (i) Show that the equation $e^{\ln x} = \sin x + 1$ has at least one solution in the interval [1,2]. (*A graph alone will not gain full marks*) [3]

(ii) Describe, with the aid of a sketch, the Secant method for solving equations. Also, point out the major difference between it and the method of false position (Linear interpolation). [4]

(iii) Using **pencil, paper and calculator** only, calculate one application of the of the secant method to approximate the solution of $e^{\ln x} = \sin x + 1$, with starting values $x_0 = 1, x_1 = 2$. [4]

(iv) Using DERIVE, or otherwise, use the secant method to find the solution to the equation $e^{\ln x} = \sin x + 1$, correct to 4 decimal places. Confirm that this solution is correct to 4 decimal places. [2]

(b) (i) The equation $x - e^{-2x} = 0$ is to be solved using the Fixed Point Iteration method. A simple rearrangement of the form $x = g(x)$ is $x = e^{-2x}$. Given that the solution to this equation is very close to $x = 0.56$, show that the above iterative scheme should converge to a solution. [3]

(ii) If one were to produce a cobweb/staircase diagram for the above iterative scheme, state with a reason whether it would be a cobweb or a staircase. (*Just producing a picture is not enough*!) [3]

(iii) Use the DERIVE ITERATES command to produce a series of iterates for the iterative scheme, $x_{n+1} = e^{-x_n}$, with a starting iterate $x_0 = 1$.

Use these iterates to determine the order of convergence and also their asymptotic error constant *C*. [6]