Solving Non Linear Equations using the Newton Raphson Method

Before we look at this we need to look at Taylor's expansion in more than one variable. Specifically we will look at two variables i.e. the expansion of $f(x, y)$ about the point (x_0, y_0).

Taylor Expansion in 2 Variables

With one variable, Taylor's expansion about the point x_0 is generated from the expression $a + b(x - x_0) + c(x - x_0)^2 + \cdots$ up to the quadratic terms.

In two variables the Taylor expansion about the point generated from (x_0, y_0) the expression

$$f(x,y) = \overbrace{a + b(x - x_0) + c(y - y_0)}^{\text{linear terms}} + \overbrace{d(x - x_0)^2 + e(x - x_0)(y - y_0) + g(y - y_0)^2}^{\text{quadratic terms}} + \cdots$$

up to the quadratic terms. Again, echoing what we did with the one variable case

$$a = f(x_0, y_0)$$

If we differentiate (partially) through wrt *x* we get

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = b + 2(x - x_0)d + e(y - y_0) + \cdots$$

so

$$\frac{\partial}{\partial x} \quad x\;, y\;) \quad f\;(x\;, y\;) \quad b \quad (x \quad x\;)d \quad e(y \quad y\;)$$

$$b = f_x(x_0, y_0)$$

By similar approach, i.e. differentiating through wrt to *y* we have

$c = f_y(x_0, y_0)$ also we can show that

$$d = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x_0, y_0) = \frac{1}{2} f_{xx}(x_0, y_0)$$

$$e = \frac{\partial^2}{\partial x \partial y} f(x_0, y_0) = f_{xy}(x_0, y_0)$$

$$g = \frac{1}{2}\frac{\partial^2}{\partial y^2} f(x_0, y_0) = \frac{1}{2} f_{yy}(x_0, y_0)$$

Leading to

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\left(x - x_0\right) + f_y(x_0,y_0)\left(y - y_0\right) +$$
$$\frac{1}{2!}\left\{f_{xx}(x_0,y_0)\left(x - x_0\right)^2 + 2f_{xy}(x_0,y_0)\left(x - x_0\right)\left(y - y_0\right) + f_{yy}(x_0,y_0)\left(y - y_0\right)^2\right\} + \cdots$$

Exercise

Find the quadratic Taylor expansion of the function $z = \sin(xy)$ about the point $(0,0)$.

$f(0,0) = 0$

$f_x(x,y) = y\cos(xy)$ $\quad$ $f_x(0,0) = 0$

$f_y(x,y) = x\cos(xy)$ $\quad$ $f_y(0,0) = 0$

$f_{xx}(x,y) = -y^2\sin(xy)$ $\quad$ $f_{xx}(0,0) = 0$

$f_{yy}(x,y) = -x^2\sin(xy)$ $\quad$ $f_{yy}(0,0) = 0$

$f_{xy}(x,y) = -x^2\sin(xy) + \cos(xy)$ $\quad$ $f_{xy}(0,0) = 1$

$$\sin(xy) = 0 + 0x + 0y + \frac{1}{2!}\left(0x^2 + 2.1.xy + 0y^2\right) + \cdots = xy + \cdots$$

As the Taylor expansion of $\sin x = x - \frac{x^3}{3!} + \cdots$ it is not surprising that $\sin(xy) \approx xy$.

The Newton Raphson Method in 2 variables

Before we study this in two variables, it is useful to see a derivation of the Newton Raphson process from the Taylor expansion with one variable.

The Taylor expansion with on variable is

$$f(x) = f(a) + f'(a)(x - a) + \cdots$$

if we expand $f(l)$, where l is a solution of the equation $f(x) = 0$, about a point near l, say x_0 then we have

$$f(l) = f(x_0) + f'(x_0)(l - x_0) + \cdots$$

If we truncate the Taylor expansion we can no longer use l on the right hand side, but some other approximation to l, namely x_1, which is (hopefully) closer to l than x_0

So $f(l) = 0 = f(x_0) + f'(x_0)(x_1 - x_0)$, solving for x_1 we get

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ which is, of course, the Newton Raphson process.

This of course leads the way to an even better iterative procedure than the Newton Raphson method by using $f(l) = f(x_0) + f'(x_0)(l - x_0) + \frac{f''(x_0)}{2!}(l - x_0)^2 + \cdots$

I leave this to you as an investigation.

If we have two non-linear equations in two unknowns say

$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}$$ which have the solutions $x = l$, $y = m$

then using the Taylor expansions of f and g about the point (x_0, y_0) near (l, m), we have

$$\begin{aligned} f(l,m) &= f(x_0,y_0) + f_x(x_0,y_0)(l - x_0) + f_y(x_0,y_0)(m - y_0) + \cdots \\ g(l,m) &= g(x_0,y_0) + g_x(x_0,y_0)(l - x_0) + g_y(x_0,y_0)(m - y_0) + \cdots \end{aligned}$$

Truncating the Taylor expansions we change x_1 with l and y_1 with m, we get

$$\begin{aligned} 0 &= f(x_0,y_0) + f_x(x_0,y_0)(x_1 - x_0) + f_y(x_0,y_0)(y_1 - y_0) \\ 0 &= g(x_0,y_0) + g_x(x_0,y_0)(x_1 - x_0) + g_y(x_0,y_0)(y_1 - y_0) \end{aligned} \qquad (*)$$

Solving these simultaneously for x_1 and y_1, we will get two iterative schemes that then use to find the two solutions. We will use a matrix approach to solve these equations for x_1 and y_1.

Rewriting (*) as

$$\begin{aligned} f_x(x_0,y_0)(x_1 - x_0) + f_y(x_0,y_0)(y_1 - y_0) &= -f(x_0,y_0) \\ g_x(x_0,y_0)(x_1 - x_0) + g_y(x_0,y_0)(y_1 - y_0) &= -g(x_0,y_0) \end{aligned}$$

In matrix form is,

$$\begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = -\begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix}$$

The matrix $\begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix}$ is called the **Jacobian** matrix $\mathbf{J_0}$

Then (*) becomes

$$\mathbf{J}_0\begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = -\begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$$

$$\begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = -\mathbf{J}_0^{-1}\begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \mathbf{J}_0^{-1}\begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$$

$$\underline{\mathbf{x}}_1 = \underline{\mathbf{x}}_0 - \mathbf{J}_0^{-1}\mathbf{F}(\underline{\mathbf{x}}_0)$$

So in general the iterative scheme for the Newton Raphson method is

$$\underline{\mathbf{x}}_{i+1} = \underline{\mathbf{x}}_i - \mathbf{J}_i^{-1}\mathbf{F}(\underline{\mathbf{x}}_i)$$

Derive Activity 9.1

Construct the iterative scheme for the Newton Raphson process for the equations

$$x^2 + y^2 = 1$$
$$y = x$$

The function Jacobian below will help us to construct the Newton raphson iterative scheme.

```
#1:    JACOBIAN_AUX(vect, vars) :=
       VECTOR(VECTOR(DIF(vectsubs,varssubr),r,1,DIMENSION(vars)),s,1,
       DIMENSION(vect))

#2:    JACOBIAN(vect) := JACOBIAN_AUX(vect, VARIABLES(vect))
```

We can test the function, where `vect` is a vector of the equations, so `JACOBIAN([f(x,y),g(x,y)])` simplified gives the Jacobian for a 2x2 set of equations. e.g

```
                 „ 2      2               †
#3:    JACOBIAN(…x    + y    - 1,  y  -  x‡)

       „  2·x    2·y  †
#4:    ¦               ¦
       …  -1       1   ‡
```

Now the iterative scheme is

$$\underline{\mathbf{x}}_{i+1} = \underline{\mathbf{x}}_i - \mathbf{J}_i^{-1}\mathbf{F}(\underline{\mathbf{x}}_i)$$

which transcribes in Derive as

```
#5:    [x,y]-Jacobian([x^2+y^2-1,y-x])^-1[x^2+y^2-1,y-x]
```

which simplifies to

$$\#6: \quad \left[\frac{2 \cdot y^2 + 1}{2 \cdot (x + y)} + \frac{x}{2} - \frac{y}{2}, \; \frac{2 \cdot y^2 + 1}{2 \cdot (x + y)} + \frac{x}{2} - \frac{y}{2} \right]$$

Which is the Newton Raphson iterative scheme for the given system of equations.
We now iterate this vector 5 times starting at [1,1]

```
#7:    ITERATES(#6,[X,Y],[1,1],5)
```

Approximating this gives the matrix of iterates for x and y

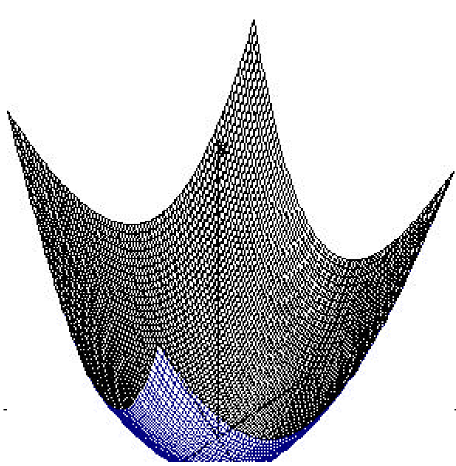
$$\#8: \quad \begin{bmatrix} 1 & 1 \\ 0.75 & 0.75 \\ 0.7083333333 & 0.7083333333 \\ 0.7071078431 & 0.7071078431 \\ 0.7071067811 & 0.7071067811 \\ 0.7071067811 & 0.7071067811 \end{bmatrix}$$

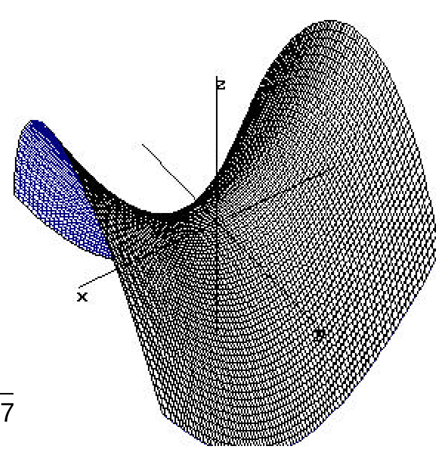
The following functions automate the above calculations

```
ITSCHM(vect) := VARIABLES(vect) - JACOBIAN(vect)^-1 ·vect

NEWTON_AUX(vect, iter, init, numb) :=
ITERATES(iter, VARIABLES(vect), init, numb)

NEWTON(vect, init, numb) :=
NEWTON_AUX(vect, ITSCHM(vect), init, numb)
```

Optimising a 2 dimensional function

If we have a 2 dimensional function, say $z = f(x, y)$, then we can find stationary points i.e. local
gradient in the x y directions are zero. Examples below





$$\frac{\partial}{\partial x}f(x,y)=0,\ \frac{\partial}{\partial y}f(x,y)=0$$

So the Jacobian of these two equations is

$$\begin{pmatrix}\frac{\partial^2}{\partial x^2}f(x,y) & \frac{\partial^2}{\partial y\partial x}f(x,y)\\ \frac{\partial^2}{\partial x\partial y}f(x,y) & \frac{\partial^2}{\partial y^2}f(x,y)\end{pmatrix}=\begin{pmatrix}f_{xx}(x,y) & f_{yx}(x,y)\\ f_{xy}(x,y) & f_{yy}(x,y)\end{pmatrix}$$

This matrix is called the **Hessian** matrix of the function $f(x,y)$, this is just the multi-dimensional second derivative.

If we wish to maximise or minimise a function $f(x,y)$, we can use the Newton Raphson method solving the with equations $\begin{pmatrix}f_x(x,y)\\ f_y(x,y)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ then the iterative scheme is modified to

$$\underline{\mathbf{x}}_{i+1}=\underline{\mathbf{x}}_i-\mathbf{H}_i^{-1}\mathbf{grad}(f(x,y))$$

where $\mathbf{grad}(f(x,y))=\begin{pmatrix}f_x(x,y)\\ f_y(x,y)\end{pmatrix}$

Example

Find the Hessian and **grad** of $z=x^3+y^3-xy^2$, hence determine the N-R iterative scheme for the extremum (turning points) of this function.

The Hessian is $\begin{pmatrix}6x & -2y\\ -2y & 6y-2x\end{pmatrix}$ and $\mathbf{grad}\left(x^3+y^3-xy^2\right)=\begin{pmatrix}3x^2-y^2\\ 3y^2-2xy\end{pmatrix}$

So the N-R iterative scheme requires the inverse of the Hessian which is

$$\begin{pmatrix}\frac{x-3\cdot y}{2\cdot(3\cdot x^2-9\cdot x\cdot y+y^2)} & -\frac{y}{2\cdot(3\cdot x^2-9\cdot x\cdot y+y^2)}\\ -\frac{y}{2\cdot(3\cdot x^2-9\cdot x\cdot y+y^2)} & -\frac{3\cdot x}{2\cdot(3\cdot x^2-9\cdot x\cdot y+y^2)}\end{pmatrix}$$

calculated with Derive.

Hence the iterative scheme is

$$[x,\ y] - \begin{bmatrix} \dfrac{x - 3\cdot y}{2\cdot(3\cdot x^2 - 9\cdot x\cdot y + y^2)} & -\dfrac{y}{2\cdot(3\cdot x^2 - 9\cdot x\cdot y + y^2)} \\ -\dfrac{y}{2\cdot(3\cdot x^2 - 9\cdot x\cdot y + y^2)} & -\dfrac{3\cdot x}{2\cdot(3\cdot x^2 - 9\cdot x\cdot y + y^2)} \end{bmatrix} \cdot \left[3\cdot x^2 - y^2,\ 3\cdot y^2 - 2\cdot x\cdot y\right]$$

which remarkable simplifies to

$$\left[\frac{x}{2},\ \frac{y}{2}\right]$$

Iterating this vector with `ITERATES([x/2,y/2],[x,y],[1,1],10)`

Starting at [1,1] leads to the (side?) saddle point at (0,0).