Mark Scheme
MA2000 1999-2000 Semester 2

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Question 1

(a)

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f(x) := 2·COS(x) – x
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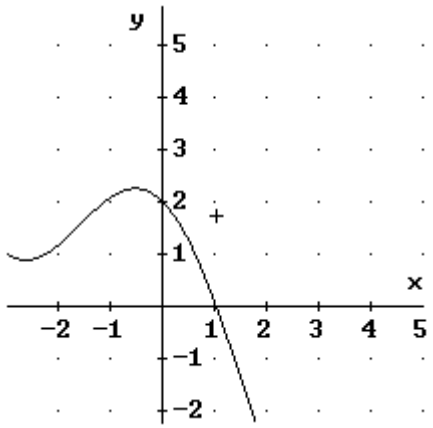


[G1]

```
[f(1), f(2)]
```

```
[0.0806046, −2.83229]
```

sign change+graphical evidence,therefore a solution

[M1A1]

(b)

- choose a starting value x0,
- draw a tangent to f(x) at (x0,f(x0))
- next iterate is the intersection of the tangent with the x axis

· (1 each)[B3]

(c)

- an initial point is near a turning point
- tangent intersects axis at a point where f(x) is undefined
- anything sensible I haven't thought of

· (any 2) [B2]

(d)

$$x - \frac{f(x)}{\frac{d}{dx} f(x)} = \frac{2\cdot(\cos(x) + x\cdot\sin(x))}{2\cdot\sin(x) + 1}$$

[M2A2]

(e)

$$\text{ITERATES}\left(\frac{2\cdot(\cos(x)+x\cdot\sin(x))}{2\cdot\sin(x)+1}, x, 1, 4\right)$$

[1, 1.030043367, 1.029866535, 1.029866529, 1.029866529]

[M1, A3]

(f)

solutions=1.02987 to 6 sig. figs

[A1]

[f(1.029865), f(1.029875)]

$$\left[4.151282993\cdot10^{-6}, -2.29933589\cdot10^{-5}\right]$$

sign change therefore 1.02987 correct to 6 sig figs

[M2,A1]

(g)

$$\text{ITERATES}\left(\frac{2\cdot(\cos(x)+x\cdot\sin(x))}{2\cdot\sin(x)+1}, x, 5.8, 10\right)$$

[5.8, −51.109732, −157.0573685, −4.780562631, −3.139085833, −1.994253947,

−3.418226345, −2.451838248, −5.785349853, −1.927347496, −3.333480303]

[M1,A1]

- No convergence after 10 iterations
- Starting value near a turning point
- hence the first iterate far away from starting value; method unstable in this case

· (1 each)[B3]

Question 2

(a)

- Use Euler's method to approximate y1 with Y1
- Use x1 and Y1 to estimate the gradient at (x1,y1)
- Calculate the average of gradient at (x0,y0) and at (x1,Y1)
- Use this average gradient instead of f(x0,y0)
- Y1=y0+h/2(f(x0,y0)+f(x0,y0+hf(x0,y0)))

· (1 each)[B5]

(b)

$$f(x, y) := x^2 + \text{SIN}(y^2)$$

[x0 := 1, y0 := 0]

Y1 := y0 + 0.2·f(x0, y0)

$$0.2$$

[M1,A1]

$$y0 + \frac{0.2}{2}\cdot(f(x0,\ y0) + f(1.2,\ Y1))$$

$$y0 + \frac{0.2}{2}\cdot(1 + 1.479989334)$$

$$0.2479989334$$

[M2,A1]

(c)

$$\text{ITERATES}\left(\left[x + 0.1,\ y + \frac{0.1}{2}\cdot(f(x,\ y) + f(x + 0.1,\ y + 0.1\cdot f(x,\ y)))\right],\ [x,\ y],\ [1,\ 0],\ 2\right)$$

$$\begin{bmatrix} 1 & 0 \\ 1.1 & 0.110999 \\ 1.2 & 0.246834 \end{bmatrix}$$

[M2,A1]

(d)

$$\text{SOLUTIONS}\left(\left[Y1 = 0.247998 + c\cdot 0.2^2,\ Y1 = 0.246834 + c\cdot 0.1^2\right],\ [Y1,\ c]\right)$$

$$[[0.246446,\ -0.0388129]]$$

answer=0.2464464148 using extrapolation

[M3,A1]

(e)

$$f1(x,\ y) := x^2 + \text{SIN}(y^2)$$

$$f2(x,\ y) := 2\cdot x + 2\cdot y\cdot\text{COS}(y^2)\cdot f1(x,\ y)$$

$$f2(x,\ y) := 2\cdot x + 2\cdot y\cdot\text{COS}(y^2)\cdot(\text{SIN}(y^2) + x^2)$$

$$f2(x,\ y) := \text{COS}(y^2)\cdot(2\cdot y\cdot\text{SIN}(y^2) + 2\cdot x^2\cdot y) + 2\cdot x$$

$$f2(x,\ y) := 2\cdot y\cdot\text{SIN}(y^2)\cdot\text{COS}(y^2) + 2\cdot x^2\cdot y\cdot\text{COS}(y^2) + 2\cdot x$$

$$y + h\cdot f1(x,\ y) + \frac{h^2}{2!}\cdot f2(x,\ y)$$

$$2 \quad 2 \quad \frac{h^2}{} \quad 2 \quad 2 \quad 2 \quad 2$$

$$y + h\cdot(\mathrm{SIN}(y) + x) + \frac{}{2!}\cdot(2\cdot y\cdot\mathrm{SIN}(y)\cdot\mathrm{COS}(y) + 2\cdot x\cdot y\cdot\mathrm{COS}(y) + 2\cdot x)$$

[M2A3]

(f)

$$0 + 0.2\cdot(\mathrm{SIN}(0^2) + 1^2) + \frac{0.2^2}{2!}\cdot(2\cdot 0\cdot\mathrm{SIN}(0^2)\cdot\mathrm{COS}(0^2) + 2\cdot 1^2\cdot 0\cdot\mathrm{COS}(0^2) + 2\cdot 1)$$

$$0.24$$

[M2A1]

Question 3

(a)

- a sketch of M rule
- M=hf(a+h/2)
- description
- a sketch of T rule
- T=h/2[f(a)+f(a+h)]
- description

. [B6]

(b)

$$\mathrm{TAYLOR}(f(x),\ x,\ a,\ 3)$$

$$\frac{(x-a)^3\cdot f'''(a)}{6} + \frac{(x-a)^2\cdot f''(a)}{2} + (x-a)\cdot f'(a) + f(a)$$

$$\int_a^{a+h}\left(\frac{(x-a)^3\cdot f'''(a)}{6} + \frac{(x-a)^2\cdot f''(a)}{2} + (x-a)\cdot f'(a) + f(a)\right)dx$$

$$\frac{h^4\cdot f'''(a)}{24} + \frac{h^3\cdot f''(a)}{6} + \frac{h^2\cdot f'(a)}{2} + h\cdot f(a)$$

[M3,A2]

(c)

$$\mathrm{TAYLOR}(f(x),\ x,\ a,\ 3)$$

$$\frac{(x-a)^3\cdot f'''(a)}{6} + \frac{(x-a)^2\cdot f''(a)}{2} + (x-a)\cdot f'(a) + f(a)$$

$$\frac{\left(\left(a + \frac{h}{2}\right) - a\right)^3 \cdot f'''(a)}{6} + \frac{\left(\left(a + \frac{h}{2}\right) - a\right)^2 \cdot f''(a)}{2} + \left(\left(a + \frac{h}{2}\right) - a\right) \cdot f'(a) +$$

$$f(a)$$

$$\frac{h^3 \cdot f'''(a)}{48} + \frac{h^2 \cdot f''(a)}{8} + \frac{h \cdot f'(a)}{2} + f(a)$$

i.e. the taylor expansion of f(a+h/2) so the Taylor expansionof hf(a+h/2) =

$$h \cdot \left(\frac{h^3 \cdot f'''(a)}{48} + \frac{h^2 \cdot f''(a)}{8} + \frac{h \cdot f'(a)}{2} + f(a) \right)$$

$$\frac{h^4 \cdot f'''(a)}{48} + \frac{h^3 \cdot f''(a)}{8} + \frac{h^2 \cdot f'(a)}{2} + h \cdot f(a)$$

subtracting the two Taylor series

$$\left(\frac{h^4 \cdot f'''(a)}{24} + \frac{h^3 \cdot f''(a)}{6} + \frac{h^2 \cdot f'(a)}{2} + h \cdot f(a) \right) - \left(\frac{h^4 \cdot f'''(a)}{48} + \frac{h^3 \cdot f''(a)}{8} + \frac{h^2 \cdot f'(a)}{2} + h \cdot f(a) \right)$$

$$\frac{h^4 \cdot f'''(a)}{48} + \frac{h^3 \cdot f''(a)}{24}$$

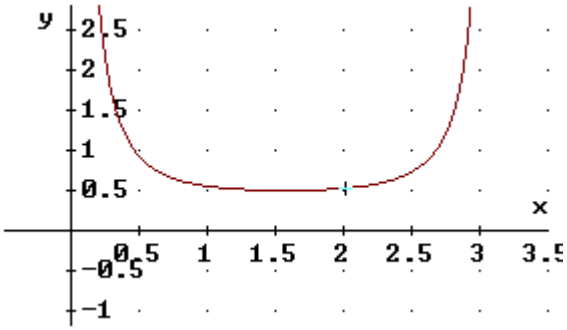
[M2,A3]

(d)

$$f(x) := \sqrt{(SIN(x))}$$

$$\left| \left(\frac{d}{dx} \right)^2 f(x) \right|$$

$$SIGN(SIN(x)) \cdot \left(\frac{COS(x)^2}{4 \cdot SIN(x)^{3/2}} + \frac{\sqrt{(SIN(x))}}{2} \right)$$



So the max in the range [1,2] is either at x=1 or x=2

$$\left[\text{SUBST}\left[\text{SIGN}(\text{SIN}(x))^{3/2}\cdot\left(\frac{\text{COS}(x)^2}{4\cdot\text{SIN}(x)^{3/2}}+\frac{\sqrt{(\text{SIN}(x))}}{2}\right),\ x,\ 1\right],\ \text{SUBST}\left[\text{SIGN}(\text{SIN}(x))^{3/2}\cdot\left(\frac{\text{COS}(x)^2}{4\cdot\text{SIN}(x)^{3/2}}+\frac{\sqrt{(\text{SIN}(x))}}{2}\right),\ x,\ 2\right]\right]$$

$$[0.553207,\ 0.526716]$$

so the max is at x=1 and M=0.5532072009

[M3,A3]

therefore |E| is

$$\frac{\left(\frac{1}{10}\right)^2\cdot(2-1)\cdot 0.553207}{24}$$

[M2,A1]

$$0.000230503$$

Question 4

(a)

$$f(x,y) := (x+2\cdot\pi)\cdot\text{COS}(x)^2 + y^2\cdot\text{SIN}(y)^2 - 1$$

$$g(x,y) := x^2 + y^2 - 4$$

$$J(x,y) := [\text{GRAD}(f(x,y),[x,y]),\ \text{GRAD}(g(x,y),[x,y])]$$

$$J(x,y) = \begin{bmatrix} \text{COS}(x)^2 - 2\cdot(x+2\cdot\pi)\cdot\text{SIN}(x)\cdot\text{COS}(x) & 2\cdot y^2\cdot\text{SIN}(y)\cdot\text{COS}(y) + 2\cdot y\cdot\text{SIN}(y)^2 \\ 2\cdot x & 2\cdot y \end{bmatrix}$$

[M3,A3]

(b)

$$[x, y] - J(x, y)^{-1} \cdot [f(x, y), g(x, y)]$$

Iterative scheme is (ouch) [**students are told not to write this down**]

$$\begin{bmatrix} - & \end{bmatrix}$$

$$\frac{2 \cdot \pi \cdot \cos(x)^2 + 2 \cdot x \cdot (x + 2 \cdot \pi) \cdot \sin(x) \cdot \cos(x) + y \cdot (x^2 - y^2 + 4) \cdot \sin(y) \cdot \cos(y) + (x^2 + 4) \cdot \sin(y)^2 - 1}{\cos(x)^2 - 2 \cdot (x + 2 \cdot \pi) \cdot \sin(x) \cdot \cos(x) - 2 \cdot x \cdot \sin(y) \cdot (y \cdot \cos(y) + \sin(y))},$$

$$\frac{(x^2 + 4 \cdot \pi \cdot x + y^2 + 4) \cdot \cos(x)^2 + 2 \cdot (x + 2 \cdot \pi) \cdot (x^2 - y^2 - 4) \cdot \sin(x) \cdot \cos(x) - 2 \cdot x \cdot (2 \cdot y \cdot (\cos(x)^2 - 2 \cdot (x + 2 \cdot \pi) \cdot \sin(x) \cdot \cos(x) - 2 \cdot x \cdot \sin(y) \cdot (y \cdot \cos(y) + \sin(y))}{2 \cdot y^3 \cdot \sin(y) \cdot \cos(y) + y^2 \cdot \sin(y)^2 + 1)}$$

$$\text{iter}(x, y) := [x, y] - J(x, y)^{-1} \cdot [f(x, y), g(x, y)]$$

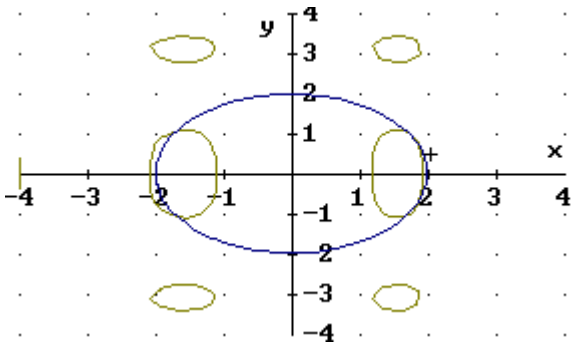
Evaluate Iter(2,1)

$$\text{iter}(2, 1)$$

$$[2.01126, \ 0.477463]$$

[M4,A3]

(c)



$$\text{ITERATES}(\text{iter}(x, y), [x, y], [2, 1], 6)$$

$$\begin{bmatrix} 2 & 1 \\ 2.01126 & 0.477463 \end{bmatrix}$$

```
[ 1.92129   0.570417 ]
[ 1.90521   0.609903 ]
[ 1.90414   0.611749 ]
[ 1.90414   0.611757 ]
[ 1.90414   0.611757 ]
```

therefore the solution near [2,1] is x=1.90414 y=0.611757

[M4,A3]

(d)

```
ITERATES(iter(x, y), [x, y], [2, -1], 6)
```

```
[    2         -1      ]
[ 2.01126   -0.477463 ]
[ 1.92129   -0.570417 ]
[ 1.90521   -0.609903 ]
[ 1.90414   -0.611749 ]
[ 1.90414   -0.611757 ]
[ 1.90414   -0.611757 ]
```

```
ITERATES(iter(x, y), [x, y], [1, -1], 6)
```

```
[    1         -1     ]
[ 1.48054   -1.51945 ]
[ 1.66159   -1.17826 ]
[ 1.68354   -1.08399 ]
[ 1.68662   -1.07488 ]
[ 1.68668   -1.07475 ]
[ 1.68668   -1.07475 ]
```

```
ITERATES(iter(x, y), [x, y], [1, 1], 6)
```

```
[    1         1     ]
[ 1.48054   1.51945 ]
[ 1.66159   1.17826 ]
[ 1.68354   1.08399 ]
[ 1.68662   1.07488 ]
[ 1.68668   1.07475 ]
[ 1.68668   1.07475 ]
```

```
[ 1.90414   0.611757  ]
[ 1.90414   -0.611757 ]
[ 1.68668   -1.07475  ]
```

```
[ 1.68668   1.07475 ]
```

[M1,A4]

Question 5

(a)

Description of shooting method (1 mark for each sensible point)

[B5]

(b)

$$\frac{d}{dx} y = v$$

$$\frac{d}{dx} v = \cos(x + y) + 2 \cdot x \cdot v^2 - e^{x \cdot y}$$

[M2,A3]

(c)

```
LOAD(C:\Maths\DfW5\MATH\Ode_appr.mth)
```

```
RK([v, COS(x + y) + 2·x·v^2 - ê^(x·y)], [x, y, v], [0, 1, 0], 0.1, 10)
```

0	1	0
0.1	0.99739	−0.0553386
0.2	0.988295	−0.12973
0.3	0.970827	−0.222531
0.4	0.943264	−0.330912
0.5	0.904312	−0.449121
0.6	0.853401	−0.568427
0.7	0.790942	−0.678267
0.8	0.718401	−0.768597
0.9	0.638107	−0.83251
1	0.552855	−0.867792

```
RK([v, COS(x + y) + 2·x·v^2 - ê^(x·y)], [x, y, v], [0, 1, −1], 0.1, 10)
```

0	1	−1
0.1	0.897888	−1.04015
0.2	0.792457	−1.06573
0.3	0.685336	−1.07357
0.4	0.578372	−1.0626
0.5	0.473403	−1.03404

$$\begin{bmatrix} 0.6 & 0.37205 & -0.990885 \\ 0.7 & 0.27558 & -0.937086 \\ 0.8 & 0.18485 & -0.87674 \\ 0.9 & 0.100327 & -0.813487 \\ 1 & 0.0221515 & -0.750233 \end{bmatrix}$$

$$RK(\begin{bmatrix} v, & \cos(x + y) + 2 \cdot x \cdot v^2 - e^{x \cdot y} \end{bmatrix}, [x, y, v], [0, 1, -1.1], 0.1, 10)$$

$$\begin{bmatrix} 0 & 1 & -1.1 \\ 0.1 & 0.887974 & -1.13757 \\ 0.2 & 0.773144 & -1.15548 \\ 0.3 & 0.657617 & -1.15132 \\ 0.4 & 0.543602 & -1.12555 \\ 0.5 & 0.433127 & -1.08116 \\ 0.6 & 0.327835 & -1.02274 \\ 0.7 & 0.228876 & -0.955347 \\ 0.8 & 0.136909 & -0.883609 \\ 0.9 & 0.0521727 & -0.811269 \\ 1 & -0.0254168 & -0.741047 \end{bmatrix}$$

$$RK(\begin{bmatrix} v, & \cos(x + y) + 2 \cdot x \cdot v^2 - e^{x \cdot y} \end{bmatrix}, [x, y, v], [0, 1, -1.09276], 0.1, 10)$$

$$\begin{bmatrix} 0 & 1 & -1.09276 \\ 0.1 & 0.888692 & -1.13052 \\ 0.2 & 0.77454 & -1.149 \\ 0.3 & 0.659619 & -1.14574 \\ 0.4 & 0.546109 & -1.12105 \\ 0.5 & 0.436028 & -1.07781 \\ 0.6 & 0.331016 & -1.02048 \\ 0.7 & 0.232233 & -0.954056 \\ 0.8 & 0.140353 & -0.883125 \\ 0.9 & 0.055632 & -0.811426 \\ 1 & -0.0219988 & -0.741696 \end{bmatrix}$$

extroploate

$$\frac{0.0221515}{0.0221515 - -0.0254168} \cdot 0.1$$

$$\frac{0.0465677}{-1 - 0.0465677}$$

The initial v should then be

$$-1.04656$$

[M6,A6]

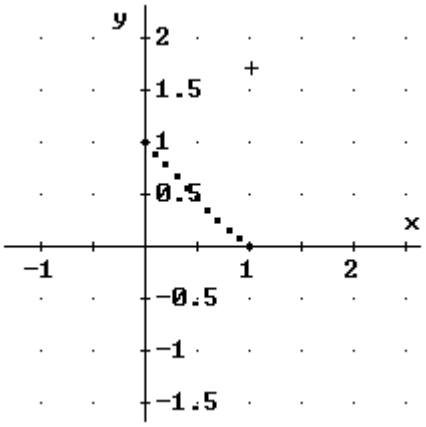
$$\mathrm{RK}\left(\left[v,\ \cos(x+y)+2\cdot x\cdot v^{2}-e^{x\cdot y}\right],\ [x,\ y,\ v],\ [0,\ 1,\ -1.04656],\ 0.1,\ 10\right)$$

$$\begin{bmatrix} 0 & 1 & -1.04656 \\ 0.1 & 0.893271 & -1.08554 \\ 0.2 & 0.783457 & -1.10761 \\ 0.3 & 0.67241 & -1.10993 \\ 0.4 & 0.562145 & -1.09211 \\ 0.5 & 0.454592 & -1.05619 \\ 0.6 & 0.351385 & -1.00589 \\ 0.7 & 0.253741 & -0.945704 \\ 0.8 & 0.162427 & -0.879987 \\ 0.9 & 0.0778035 & -0.812441 \\ 1 & -9.56638\cdot 10^{-5} & -0.745895 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 & -1.04656 \\ 0.1 & 0.893271 & -1.08554 \\ 0.2 & 0.783457 & -1.10761 \\ 0.3 & 0.67241 & -1.10993 \\ 0.4 & 0.562145 & -1.09211 \\ 0.5 & 0.454592 & -1.05619 \\ 0.6 & 0.351385 & -1.00589 \\ 0.7 & 0.253741 & -0.945704 \\ 0.8 & 0.162427 & -0.879987 \\ 0.9 & 0.0778035 & -0.812441 \\ 1 & -9.56638\cdot 10^{-5} & -0.745895 \end{bmatrix} \\ \text{COL } 1, \\ \begin{bmatrix} 0 & 1 & -1.04656 \\ 0.1 & 0.893271 & -1.08554 \\ 0.2 & 0.783457 & -1.10761 \\ 0.3 & 0.67241 & -1.10993 \\ 0.4 & 0.562145 & -1.09211 \\ 0.5 & 0.454592 & -1.05619 \\ 0.6 & 0.351385 & -1.00589 \\ 0.7 & 0.253741 & -0.945704 \\ 0.8 & 0.162427 & -0.879987 \\ 0.9 & 0.0778035 & -0.812441 \\ 1 & -9.56638\cdot 10^{-5} & -0.745895 \end{bmatrix} \end{bmatrix} \text{COL}$$

$$2$$

$$\begin{bmatrix} 0 & 1 \\ 0.1 & 0.893271 \\ 0.2 & 0.783457 \\ 0.3 & 0.67241 \\ 0.4 & 0.562145 \\ 0.5 & 0.454592 \\ 0.6 & 0.351385 \\ 0.7 & 0.253741 \\ 0.8 & 0.162427 \\ 0.9 & 0.0778035 \\ 1 & -9.56638 \cdot 10^{-5} \end{bmatrix}$$



[G3]