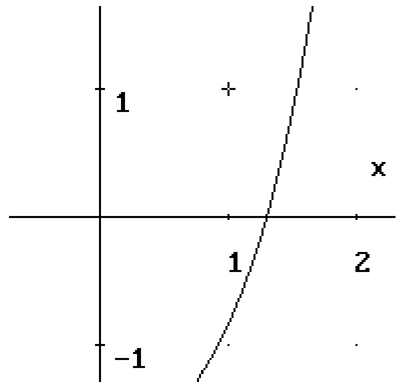
MA200 Semester 2 1996 Mark Scheme TAE

1. (i)



[G1]

$$\left.\begin{array}{l} f(1) = -0.8417 \\ f(2) = 4.2124 \end{array}\right\} \text{M1}$$ therefore a solution in [1,2] **[M1,A1]**

(ii) Graphical description of NR method **[G1]**
Written description **[B1]**
Development of iterative scheme **[M1,A1]**
Description of a failure **[B1]**

(iii)

$$\hat{e}^{x}\ \mathrm{LN}(x) - \mathrm{SIN}(x)$$

$$\frac{d}{dx}\left(\hat{e}^{x}\ \mathrm{LN}(x) - \mathrm{SIN}(x)\right)$$

$$x - \frac{\hat{e}^{x}\ \mathrm{LN}(x) - \mathrm{SIN}(x)}{\frac{d}{dx}\left(\hat{e}^{x}\ \mathrm{LN}(x) - \mathrm{SIN}(x)\right)}$$

$$\frac{x\left(\hat{e}^{x}\left((x - 1)\ \mathrm{LN}(x) + 1\right) - x\ \mathrm{COS}(x) + \mathrm{SIN}(x)\right)}{\hat{e}^{x}\left(x\ \mathrm{LN}(x) + 1\right) - x\ \mathrm{COS}(x)}$$

[M1,A1]

$$\mathrm{ITERATES}\left[\frac{x\left(\hat{e}^{x}\left((x - 1)\ \mathrm{LN}(x) + 1\right) - x\ \mathrm{COS}(x) + \mathrm{SIN}(x)\right)}{\hat{e}^{x}\left(x\ \mathrm{LN}(x) + 1\right) - x\ \mathrm{COS}(x)},\ x,\ 1.5,\ 5\right]$$

[1.5, 1.32686, 1.30077, 1.30024, 1.30024, 1.30024]

[M1,A1]

$x = 1.3002$ to 4 d.p. **[A1]**

$f(1.30015) = -3.38\ \ 10^{-4} < 0$

$f(1.30025) = 1.366\ \ 10^{-5} > 0$ **[M1 A1]**

(b) (i) $g'(x) = \dfrac{1}{2}\cos x e^{\sin x}$ **[B1]**

$g'(1.32) \approx 0.327$

therefore in the range [-1,1] so it converges **[M1,A1]**

(ii) $\dfrac{e_4}{e_3} \approx \dfrac{\Delta x_4}{\Delta x_3} = \dfrac{x_4 - x_3}{x_3 - x_2} = \dfrac{1.30758 - 1.29178}{1.29178 - 1.25055} \approx 0.384$ **[M2,A1]**

students may use CONV prog to produce the rest

$$\text{ITERATES}\left[\frac{1}{2}\,\hat{e}^{\text{SIN(x)}}, x, 1, 6\right]$$

```
[1, 1.15988, 1.25058, 1.29178, 1.30758, 1.31312, 1.31500]
```

$$\text{CONV(x, n) := VECTOR}\left[\left|\frac{x_{r+2} - x_{r+1}}{x_{r+1} - x_r}\right|, r, 1, n-1\right]$$

```
CONV([1, 1.159888, 1.250584, 1.291780, 1.307582, 1.313127, 1.315003], 6)

[0.5672488, 0.4542168, 0.3835701, 0.3509158, 0.3383830]
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[B2]

$C \approx 0.34$ (± 0.005) **[B1]**

Linear convergence **[B1]**

2. (i) Diagram showing a single trapezia **[G1]**

Develop $\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\left(f(a)+f(b)\right)$ or

$\frac{h}{2}\left(f(a)+f(b)\right)$ **[B1]**

New Diagram for n trapzia with n strips **[G1]**

Develop the correct fomula **[M1,A1]**

(ii) $h = 0.25$ **[B1]**

$T_4 = \frac{0.25}{2}\left(0.84147 + 0.45465 + 2(0.75919 + 0.665 + 0.56228)\right)$

=0.65863 **[M2,A1]**

(iii) Any method that gives 0.659155 **[M2,A1]**

(iv) Possible solution:

$I \approx T_4 + ch^2$, **[B1]**

then halve the step size $I \approx T_8 + c\left(\frac{h}{2}\right)^2$

and eliminate ch^2 to give the result. **[M2,A2]**

$I \approx \frac{\left(4(0.659155) - 0.65863\right)}{3} = 0.65942$ **[M1,A2]**

(b)

	Sn	**R1,n**	**R2,n**
2	0.8799444	0.8819370	0.882082246
4	0.8818124	0.8820824	
8	0.8820655		

application of Richardson given **[M1,A1]**

New Richardson $\frac{64R_{2n} - R_n}{63}$ **[M1,A1]**

accurate calculation of R2,n **[A1]**

3.(a) Description should:

Make reference to a first approx of y_1 $(\bar{y}_1)$ via Euler's method. **[B1]**

A reference to, and a graph, showing that the mean of $f(x_0, y_0)$ and $f(x_1, y_1)$ would produce a better estimate of y_1. **[G1,B1]**

Points out y_1 is not known but first approx $(\bar{y}_1)$ can be used. **[B1]**

Develops the formula

$$k1 = hf(x_0, y_0)$$

$$k2 = hf(x_1, y_0 + hf(x_0, y_0))$$

$$\bar{\bar{y}}_1 = y_0 + \frac{1}{2}(k1 + k2)$$

[B2]

$$k1 = 0.1 f(1,0) = 0.1(1 + 2(0)) = 0.1$$

$$k2 = 0.1 f(1.1, 0.1) = 0.1(1.1 + 2(0.1) = 0.13$$

$$y_1 = y_0 + \frac{k1 + k2}{2} = 0 + \frac{0.23}{2} = 0.115$$

[M1,A2]

```
#2:  SRK(x, y, x0, y0, h, n) := ITERATES([x + h, y + h/2 (F(x, y) + F(x + h, y +

#3:  SRK(x, y, 1, 0, 0.05, 2)

         ⎡  1       0        ⎤
#4:      ⎢ 1.05    0.05375   ⎥
         ⎣ 1.1    0.11576875 ⎦
```

0.11576875 *same marks if they do it by hand* **[M1, A1]**

(b) Use Richardson's extraopaltion **[M1]**

Develop $\frac{(4\bar{\bar{y}}_1 - \bar{y}_1)}{3} = \frac{(4(0.11576875) - 0.115)}{3}$ **[M2,A2]**

=0.116025 **[A1]**

(c) Use the Taylor expansion

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x)$$ **[M1]**

where $y' = x + 2y$, $y'' = 1 + 2y' = 1 + 2(x + 2y)$

and $y''' = 2 + 4y' = 2 + 4(x + 2y)$ **[M1,A1]**

produce given expression **[A1]**

Sub values into given expression

evaluate 0.116000. **[A1]**

(d) In (b) h=0.05 and extrapolation is $O(h^4)$. **[B1]**

In (c) we have a single application with error $O(h^4)$ **[B1]**

but in (c) h=0.1 **[B1]**

4.(a) Description to include,

rearrangement into the form $x = f(y,z), y = g(x,z)$ etc **[B1]**

use a starting value for each variable **[B1]**

iterate until convergent **[B1]**

Jacobi uses the scheme $x_{n+1} = f(y_n, z_n), y_{n+1} = g(x_n, z_n), z_{n+1} = h(x_n, y_n)$ **[B1]**

G_S uses the scheme $x_{n+1} = f(y_n, z_n), y_{n+1} = g(x_{n+1}, z_n), z_{n+1} = h(x_{n+1}, y_{n+1})$ **[B1]**

(i) In the equation matrix, $|a_{11}| \geq |a_{12}|, |a_{22}| > |a_{21}|$ or vice versa. **[B1]**

(ii) $|a_{ii}| \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|$ and strictly > in at least one case **[B2]**

(b) Re arrange the equtions into the form

$$\left.\begin{array}{l} 2x + y = 2 \\ -x + 2y = 2 \end{array}\right\} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$ **[M1,A1]**

Diagonal dominance therefor will converge **[B1]**

$$\left.\begin{array}{l} x - 2y = -2 \\ 2x + y = 2 \end{array}\right\} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$ **[M1,A1]**

no diagonal dominance, may not converge **[B1]**

(c) Rearrange for diagonal dominance

$$\left.\begin{array}{l} 2x - y + z = 3 \\ x - 2y - z = -6 \\ x + y + 3z = 12 \end{array}\right\} \Rightarrow \begin{array}{l} x^{(n+1)} = \frac{1}{2}\left(3 + y^{(n)} - z^{(n)}\right) \\ y^{(n+1)} = \frac{1}{2}\left(6 + x^{(n+1)} - z^{(n)}\right) \\ z^{(n+1)} = \frac{1}{3}\left(-x^{(n+1)} - y^{(n+1)} + 12\right) \end{array}$$

[M2,A3]

$$x^{(1)} = \frac{1}{2}\left(3 + 0 - 0\right) = 1.5$$

$$y^{(1)} = \frac{1}{2}\left(6 + 1.5 - 0\right) = 3.75$$

$$z^{(1)} = \frac{1}{3}\left(-1.5 - 3.75 + 12\right) = 2.25$$

[M1,A3]

After 10 iterations [1.14788,2.13478,2.90577] **[B2]**

5.(a)

iterate	eigen value	eigen vector
1	2	[1, -0.5, 0.5]
2	-5.5	[-0.636,1,-0.2727]
3	6.0909	[-0.44776,1,-0.2537]
4	5.1764	[-0.42558,1,-0.2637]
5	5.6423	[-0.4035,1,-0.2707]
6	5.61916	[-0.3934,1,0.2743]

[M2,A6]

The inverse is $\begin{pmatrix} \frac{5}{11} & \frac{2}{11} & \frac{3}{11} \\ \frac{4}{11} & \frac{9}{11} & \frac{6}{11} \\ \frac{2}{11} & \frac{3}{11} & \frac{10}{11} \end{pmatrix}$ **[B1]**

iterate	eigen value	eigen vector
1	1.727	[0.526,1,1.3636]
2	1.38277	[0.4602,1,0.7855]
3	1.35546	[0.4465,1,0.7897]
4	1.35397	[0.44326,1,0.7916]

smallest eigen value is approx 1/1.35397=0.738567 **[M2,A4]**

(b) Simultaneous equations are

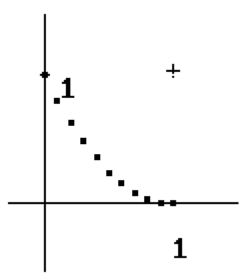
$$y' = z$$

$$z' = 2xz - xy + 3$$

[M2,A2]

Any method that gets a starting value for z to be approx -2.12

[M3,A1]



[G2]