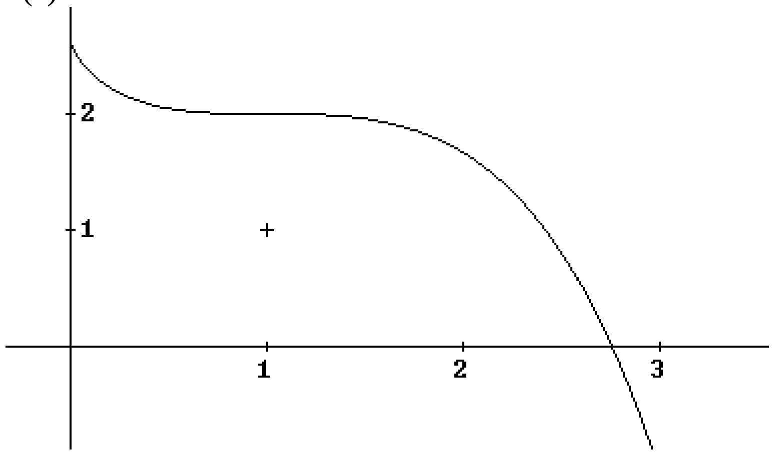
CMSMA2000 Marking Scheme 1999 Semester 2

1(a)



[G1]

$$F(x) := x \cdot LN(x) - \hat{e}^{x-1} + 3$$

[F(2), F(3)]

[1.66801253266, -1.09321923292] a solution in [2,3] [M1,A1]

(b) Bookwork,: rearrangement $|g'(l)| < 1$ for convergence [B5]

(c) Correct manipulation for both equations [A2]

(d)

$$\frac{d}{dx} \frac{\hat{e}^{x-1} - 3}{LN(x)}$$

$$\hat{e}^{x} \cdot \left(\frac{\hat{e}^{-1}}{LN(x)} - \frac{\hat{e}^{-1}}{x \cdot LN(x)^{2}} \right) + \frac{3}{x \cdot LN(x)^{2}}$$

$$\hat{e}^{2.75} \cdot \left(\frac{\hat{e}^{-1}}{LN(2.75)} - \frac{\hat{e}^{-1}}{2.75 \cdot LN(2.75)^{2}} \right) + \frac{3}{2.75 \cdot LN(2.75)^{2}}$$

4.70977832098 therefore does not converge

$$\frac{d}{dx} \hat{e}^{(\hat{e}^{x-1} - 3)/x}$$

$$\hat{e}^{\hat{e}^{x-1}/x} \cdot \left(\frac{\hat{e}^{x - 3/x - 1} \cdot (x - 1)}{x^{2}} + \frac{3 \cdot \hat{e}^{-3/x}}{x^{2}} \right)$$

$$\hat{e}^{2.75-1}/2.75 \cdot \left(\frac{\hat{e}^{2.75-3/2.75-1}\cdot(2.75-1)}{2.75^{2}} + \frac{3\cdot\hat{e}^{-3/2.75}}{2.75^{2}} \right)$$

4.70597907172 therefore does not converge

$$\frac{d}{dx}\,(\mathrm{LN}(x\cdot \mathrm{LN}(x) + 3) + 1)$$

$$\frac{\mathrm{LN}(x) + 1}{x\cdot \mathrm{LN}(x) + 3}$$

$$\frac{\mathrm{LN}(2.75) + 1}{2.75\cdot \mathrm{LN}(2.75) + 3}$$

0.347913322509 therefore does converge [M3,A4]

(e)
ITERATES(LN(x·LN(x) + 3) + 1, x, 2, 10)

[2, 2.47848476154, 2.65815002126, 2.72253233259, 2.74515449347, 2.75304628314, 2.75579235465, 2.75674704369, 2.75707884457, 2.75719414910, 2.75723421722]

Solution 2.757 to 4 sig figs [M1,A2]

Check [F(2.7565), F(2.7575)]=
[0.00285616561097, -0.000924717714981]
or an argument based on convergence [B1]

(f)

$$\mathrm{LCONV}(v) := \mathrm{VECTOR}\left(\frac{v_{r+2} - v_{r+1}}{v_{r+1} - v_{r}},\ r,\ 1,\ \mathrm{DIMENSION}(v) - 2 \right)$$

LCONV([2, 2.47848476154, 2.65815002126, 2.72253233259, 2.74515449347, 2.75304628314, 2.75579235465, 2.75674704369, 2.75707884457, 2.75719414910, 2.75723421722])

[0.375487944781, 0.358345911893, 0.351372301066, 0.348852159237, 0.347965623168, 0.347656292492, 0.347548642451, 0.347511224196, 0.347498229508]

Asymptotic error constant is approx 0.347 (as in part (d))
therefore $e_{n+1} \approx (0.347e_n)$ hence linearly convergent [M2,A2]

2. (a) Bookwork [B5]

(b)

$$F(x) :=$$

$$M(a, b, n) := \sum_{r=1}^{n} \left(\frac{b - a}{n} \cdot F\left(a + \frac{(2 \cdot r - 1) \cdot (b - a)}{2 \cdot n} \right) \right)$$

```
F(x) := SIN(‹x)

VECTOR(M(0, ¹, n), n, [10, 20, 40, 80])

[2.68029547765, 2.67331581674, 2.67084395145, 2.66997060484]
```

[M2,A3]

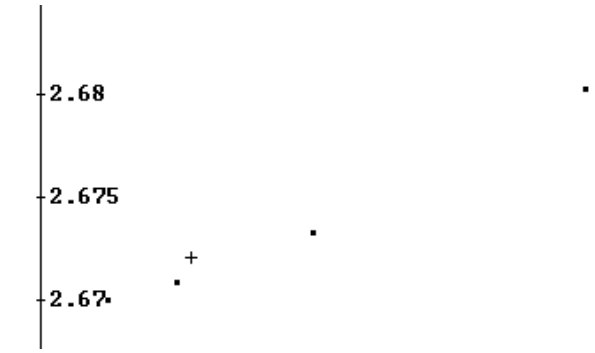
(c)

$$\text{VECTOR}\left(\left[\frac{1}{n}, M(0, 1, n) \right], n, [10, 20, 40, 80] \right)$$

$$\begin{bmatrix} 0.314159265358 & 2.68029547765 \\ 0.157079632679 & 2.67331581674 \\ 0.0785398163397 & 2.67084395145 \\ 0.0392699081698 & 2.66997060484 \end{bmatrix}$$

[M1,A2]

(d)

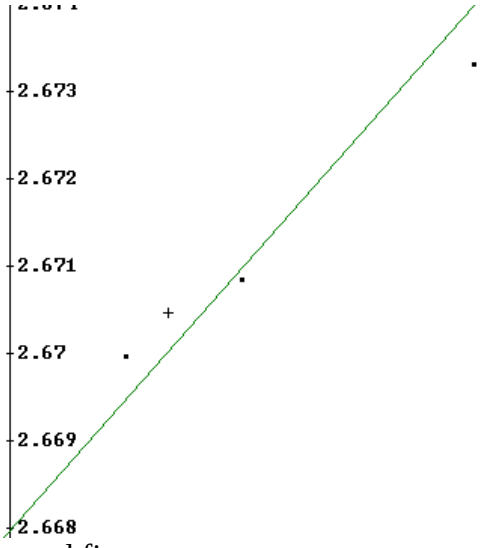


[G2]

(e)

```
    €                 „ 0.314159265358   2.68029547765 †,
    ¦                 ¦                             ¦¦
    ¦                 ¦ 0.157079632679   2.67331581674 ¦¦
FIT¦[h, a + b·h],     ¦                             ¦¦
    ¦                 ¦ 0.0785398163397  2.67084395145 ¦¦
    ¦                 ¦                             ¦¦
    •                 … 0.0392699081698  2.66997060484 ‡ƒ
```

$0.0382535683572 \cdot h + 2.66797315973$

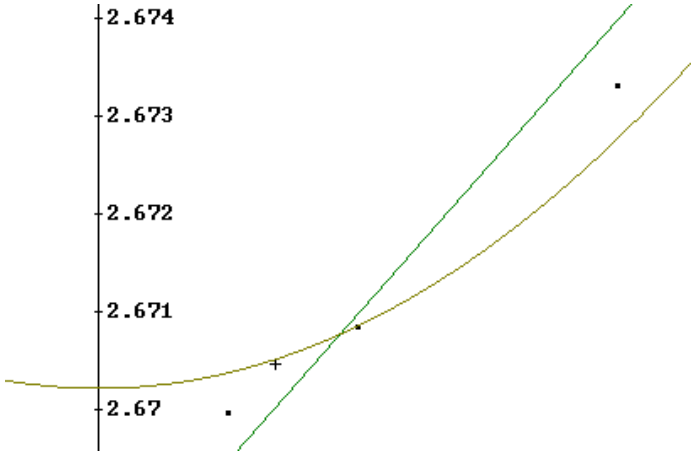


Not a good fit [M2,A1]

(f)

```
    €                 „ 0.314159265358   2.68029547765 †,
    ¦                 ¦                             ¦¦
    ¦          2†     ¦ 0.157079632679   2.67331581674 ¦¦
FIT¦…h, a + b·h ‡,     ¦                             ¦¦
    ¦                 ¦ 0.0785398163397  2.67084395145 ¦¦
    ¦                 ¦                             ¦¦
    •                 … 0.0392699081698  2.66997060484 ‡ƒ
```

$$0.103429194651 \cdot h^2 + 2.67021707028$$

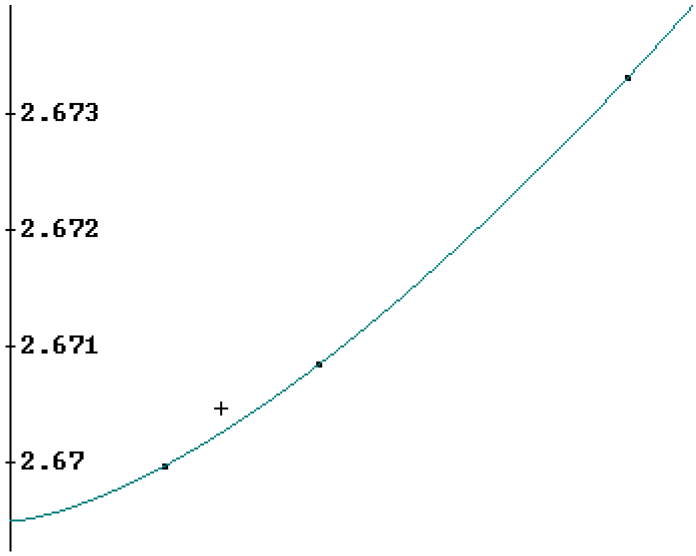


Still not good, but should be as Comp Midpoint rule is $O(h^2)$ [M1,A1]

(g)

$$\text{FIT}\left(\left[h,\ a + b\cdot h^{1.5}\right],\ \begin{bmatrix} 0.314159265358 & 2.68029547765 \\ 0.157079632679 & 2.67331581674 \\ 0.0785398163397 & 2.67084395145 \\ 0.0392699081698 & 2.66997060484 \end{bmatrix}\right)$$

$$0.0613444356094\cdot h^{1.5} + 2.66949432253$$



A near perfect fit error order 1.5 [M1,A2]

(h) Taylor expansion at 0 and π do not exist [B2]

3(a) Bookwork [B5]

(b) $f(x,y) = x^2 + \sin(y^2)$

$$y'_1 = y_0 + hf(x_0, y_0) = 0 + 0.2(1^2 + \sin(0^2)) = 0.2$$

$$y_1 = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y'_1)) =$$

$$0 + \frac{0.2}{2}\left((1^2 + \sin(0^2)) + (1.2^2 + \sin(0.2^2))\right) = 0.24799893$$

[M2,A3]

(c)

```
F(x, y) :=

MOD_EULER(x, y, x0, y0, h, n):=
ITERATES([x + h, (h·F(x + h, h·F(x, y) + y) + h·F(x, y) + 2·y)/2], [x, y], [x0, y0], n)

F(x, y) := x^2 + SIN(y^2)

MOD_EULER(x, y, 1, 0, 0.1, 2)

[ 1     0              ]
[ 1.1   0.110999991666 ]
[ 1.2   0.246834544514 ]
```

[M1,A2]

(d)

$$y_{1,1} = Y + ch^2$$

$$y_{1,2} = Y + c\left(\frac{h}{2}\right)^2$$

$$Y = (4y_{1,2} - y_{1,1})/3 =$$

$$(4 * 0.246834544514 - 0.24799893)/3 = 0.246446416018$$

[M2,A2]

(e) $$y(x+h) = y(x) + h\frac{d}{dx}y(x) + \frac{h^2}{2!}\frac{d^2}{dx^2}y(x) + \ldots\ldots$$

$$\frac{d}{dx}y(x) = x^2 + \sin(y^2)$$

$$\frac{d^2}{dx^2}y(x) = 2x + 2y\cos(y^2)\frac{dy}{dx}$$

$$= 2x + 2y\cos(y^2)\left(x^2 + \sin(y^2)\right)$$

$$= 2x + 2x^2y\cos(y^2) + 2y\cos(y^2)\sin(y^2)$$

[M1,A3]

Printed result [A1]

(f) $$y(1.2) = 0 + 0.2(1^2 + \sin(0^2)) + \frac{0.2^2}{2}(2 + 0 + 0)$$

$$= 0.2 + 0.04 = 0.24$$

[M1,A2]

4. (a)

$$J_0 = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} =$$

$$\begin{bmatrix} y \cdot \mathrm{COS}(x) + \mathrm{COS}(y) & \mathrm{SIN}(x) - x \cdot \mathrm{SIN}(y) \\ 2 \cdot x & -2 \cdot y \end{bmatrix}$$

[M3,A3]

$$[x,\ y] - \begin{bmatrix} y \cdot \mathrm{COS}(x) + \mathrm{COS}(y) & \mathrm{SIN}(x) - x \cdot \mathrm{SIN}(y) \\ 2 \cdot x & -2 \cdot y \end{bmatrix}^{-1} \cdot \left[x \cdot \mathrm{COS}(y) + y \cdot \mathrm{SIN}(x) - 1,\ x^2 - y^2 - 1 \right]$$

$$\begin{bmatrix} \dfrac{2 \cdot x \cdot y^2 \cdot \mathrm{COS}(x) + (x^2 - y^2 + 1) \cdot \mathrm{SIN}(x) - x \cdot (x^2 + y^2 + 1) \cdot \mathrm{SIN}(y) + 2 \cdot y}{2 \cdot (y^2 \cdot \mathrm{COS}(x) + x \cdot \mathrm{SIN}(x) + y \cdot \mathrm{COS}(y) - x^2 \cdot \mathrm{SIN}(y))} \\[2ex] \dfrac{y \cdot (x^2 + y^2 - 1) \cdot \mathrm{COS}(x) - (x^2 - y^2 + 1) \cdot \mathrm{COS}(y) - 2 \cdot x \cdot (x \cdot y \cdot \mathrm{SIN}(y) - 1)}{2 \cdot (y^2 \cdot \mathrm{COS}(x) + x \cdot \mathrm{SIN}(x) + y \cdot \mathrm{COS}(y) - x^2 \cdot \mathrm{SIN}(y))} \end{bmatrix}$$

[M3,A4]

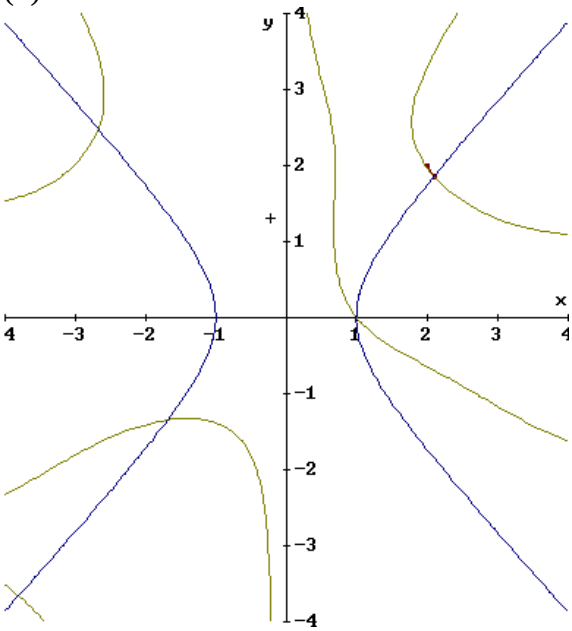
ITERATES(iter, [x, y], [2, 2], 5)

$$\begin{bmatrix} 2 & 2 \\ 2.09900439420 & 1.84900439420 \\ 2.10878058826 & 1.85658759567 \\ 2.10871005812 & 1.85651773736 \\ 2.10871005429 & 1.85651773303 \\ 2.10871005429 & 1.85651773303 \end{bmatrix}$$

[M3,A4]

(iter is the iterative scheme above)

(d)




```
ITERATES(iter, [x, y], [1, 0], 5)
```

```
„ 1  0 †
  1  0
  1  0
  1  0
  1  0
… 1  0 ‡
```

```
ITERATES(iter, [x, y], [-3, 3], 5)
```

```
„       -3              3       †
  -2.64651031843  2.47984365176
  -2.67606084897  2.48202199401
  -2.67587330257  2.48199474153
  -2.67587329519  2.48199474101
… -2.67587329442  2.48199473964 ‡
```

```
ITERATES(iter, [x, y], [-2, -2], 5)
```

```
„       -2              -2       †
  -1.58087277335  -1.33087277335
  -1.69099284770  -1.35946605547
  -1.68895088996  -1.36108543505
  -1.68895215556  -1.36108757388
… -1.68895215556  -1.36108757388 ‡
```

[M2,A3]

5.(a)
$$z' = 2xz - \sqrt{xy} + \sin x$$
$$y' = z$$

[M2,A3]

(b) Bookwork [B5]

(c) SHOOT() is a user defined function, anything similar OK

```
SHOOT(-0.6)

[1, 0.0323264028813, -1.74502590478]

SHOOT(-0.6235)

[1, 0.000831162887800, -1.79158783614]
```

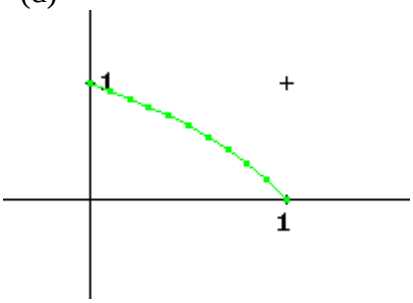
Value of $-0.623 \le y' \le -0.624$ [M3,A3]

```
RK([z, 2·x·z - ‹(x·y) + SIN(x)], [x, y, z], [0, 1, -0.6235], 0.1, 10)
```

0	1	-0.6235
0.1	0.936873180926	-0.644629047854
0.2	0.870458719339	-0.686292735148
0.3	0.799120934616	-0.742908988527
0.4	0.721373560219	-0.814698884527
0.5	0.635606987799	-0.903774650505
0.6	0.539922632354	-1.01379449618
0.7	0.431976256949	-1.15004298988
0.8	0.308805247996	-1.31963304729
0.9	0.166643084145	-1.53143677301
1	0.000831162887800	-1.79158783614

[M3,A3]

(d)



[G3]