Liverpool John Moores University
SCHOOL OF COMPUTING AND MATHEMATICAL SCIENCES

Semester 2 Examinations 2000

CMSMA2000

Numerical Methods and their Analysis

Duration 2 Hours

Instructions to candidates

Do not open this question paper until you have been told to do so by the invigilator.

The figure in [] denotes the number of marks available for that question or part of question.

There are 5 questions. **Answer 3 questions.**

Questions carry 25 marks each. The total number of marks available is 75.

You have the use of the Software DERIVE 5™. You may also use any Derive functions that you have developed throughout the course. You may not use a printer to print out any mathematical expressions or graphs.

Any results or graphs that DERIVE 5 produces, on screen, that are relevant to your examination work should be written or sketched in your answer book.

You may also use **any** calculator and its memory need **not** be erased before the examination.

The sending or the reading of email and use of the internet/intranet during this examination is prohibited.

1.

(a) Show that the equation $2\cos x = x$ has at least one solution in the interval [1,2]. (*A graph alone will not gain full marks*) [3]

(b) Describe, with the aid of a sketch, the Newton Raphson iterative method for solving equations. [3]

(c) Describe a situation in which the Newton Raphson method will fail to produce a solution for a particular equation. [2]

(d) Establish the Newton Raphson iterative formula for the equation $2\cos x = x$. [4]

(e) Using a starting value of $x_0 = 1$, find each of the Newton Raphson iterates x_1, x_2, x_3 and x_4, to 10 significant figures. [4]

(f) Write down the solution to the equation $2\cos x = x$, correct to 6 significant figures. Confirm that this solution is correct to 6 significant figures. [4]

(g) Use Newton Raphson Method to find the solution to equation $2\cos x = x$ using the starting value $x_0 = 5.8$.

Explain clearly what has happened in this particular case and give a reason for this. [5]

Total [25]

2.

(a) Describe fully, with reference to a sketch, the modified Euler's method for solving first order differential equations $\frac{dy}{dx} = f(x, y)$, with initial conditions $y(x_0) = y_0$. [5]

(b) You are given the differential equation $\frac{dy}{dx} = (x^2 + \sin\left(y^2\right))$ with initial conditions $y(1) = 0$.

Show that the modified Euler method will yield an approximate solution to the above differential equation of $y(1.2) \approx 0.2479989334$, using a step size h=0.2 (**for this part of the question you must show all the numerical steps that lead to your solution)**. [5]

(c) Find a further estimate (preferably using Derive), for $y(1.1)$, using the modified Euler method with a step size h=0.1. [3]

(d) The order of convergence of the modified Euler method is h^2. Use the order of convergence of the Euler method to extrapolate a more accurate solution of $y(1.1)$, using your answers above. [4]

(e) Use the Taylor series method of order 2 to show that the solution to the differential equation $\frac{dy}{dx} = (x^2 + \sin\left(y^2\right))$ can be approximated by

$$y(x+h) \approx y(x) + h(x^2 + \sin\left(y^2\right)) + \frac{h^2}{2!}(2x + 2x^2 y\cos(y^2) + 2y\cos(y^2)\sin(y^2))$$

where h is the step size. [5]

(f) Use the initial conditions $y(1) = 0$, a step size of 0.2 and the above Taylor expansion, to estimate $y(1.1)$. [3]

Total [25]

3.

(a) Describe, with the aid of sketches, the Midpoint rule and the Trapezium rule (applied once) for estimating the integral $\int_a^b f(x)\,dx$.

[6]

(b) Given that the first four terms of the Taylor expansion for $f(x)$ about the point $x = a$ is

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \frac{(x-a)^3}{3!}\,f'''(a) + \cdots$$

show that the first four terms of the Taylor expansion of the integral $\int_a^{a+h} f(x)\,dx$ about $x = a$ is:

$$hf(a) + \frac{h^2}{2!}\,f'(a) + \frac{h^3}{3!}\,f''(a) + \frac{h^4}{4!}\,f'''(a)$$

[5]

(c) Using the result in (b) and the Taylor expansion of $h\left(f\left(a+\frac{h}{2}\right)\right)$, show that the principal error term of the Midpoint rule is

$$\frac{h^3 f''(a)}{24}$$

[5]

(d) The upper bound of the error, E, using the **composite** trapezium rule is

$$|E| = \frac{h^2(b-a)M}{24}$$

where $M = \max|f''(x)|$ in the interval $[a,b]$.

Find an upper bound for the error in using the Midpoint rule with 10 strips to estimate

$$\int_1^2 \sqrt{\sin x}\,dx.$$

[9]

Total 25

4.

(a) The Newton Raphson iteration scheme for the solution of two simultaneous non linear equations, $f(x, y) = 0$ and $g(x, y) = 0$, is given by the expression

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \mathbf{J}_n^{-1} \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}$$

where $\mathbf{J}$ is the Jacobian matrix.

Calculate the Jacobian matrix for the 2 non-linear simultaneous equations $(x + 2\pi)\cos^2(x) + y^2 \sin^2 y = 1$ and $x^2 - y^2 = 1$. [6]

(b) Hence develop, but **do not** write down, a vector iterative scheme for the solution of the equations

$$(x + 2\pi)\cos^2(x) + y^2 \sin^2 y = 1$$

and

$$x^2 - y^2 = 1$$

Evaluate and write down, to 10 significant figures, the value of the iterative scheme when $x = 2$ and $y = 1$. [7]

(c) Use this iterative scheme to find a solution to these equations near the point $(2, 2)$. [7]

(d) Find the other three solutions to the equations

$$(x + 2\pi)\cos^2(x) + y^2 \sin^2 y = 1$$

and

$$x^2 - y^2 = 1$$

in the region $1 \le x \le 2$, $-1 \le y \le 1$.

[5]

Total [25]

5.

(a) Describe the shooting method for the solution of boundary value problems for second order ordinary differential equations. [5]

(b) Recast the second order differential equation given below into two first order differential equations $\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 + e^{xy} = \cos(x+y)$ [5]

(c) Using the RK function found in ODE_APPR.MTH file and the shooting method, numerically solve the above differential equation with the boundary values $y(0) = 1$ and $y(1) = 0$, using a step size of h=0.1. [12]

(d) Provide a sketch of the solution of $y(x)$ for $0 \le x \le 1$. [3]

Total [25]