MA1012 Solutions Semester 2 1999

1 (a) Book Work [B3]

(b) [a := [1, -1, 1], b := [1, 1, -1], c := [0, 1, 0]]

CROSS(a, b) = [0, 2, 2] [M1,A1]

CROSS(a, c) = [-1, 0, 1] [M1,A1]

CROSS(a, b + c) = [-1, 2, 3] [M1,A1]

CROSS(a, b) + CROSS(a, c) = [-1, 2, 3] [M1,A1]

CROSS(a, CROSS(b, c)) = [-1, 0, 1] [M1,A1]

(a • c)·b - (a • b)·c = [-1, 0, 1] [M1,A1]

(c) [oa := [-1, 1, 2], ob := [3, -2, 1], oc := [-2, 3, 1]]

-oa + ob = [4, -3, -1]

-oa + oc = [-1, 2, -1] [M1,A2]

CROSS(-oa + ob, -oa + oc) = [5, 5, 5] [M1,A1]

r :- Nonscalar

(r - oa) • [1, 1, 1] = 0

r • [1, 1, 1] - 2 = 0 [M2,A3]

[x, y, z] • [1, 1, 1] - 2 = 0

x + y + z - 2 = 0 [M1,A1]

Total [25]

2 (a)

$$\mathrm{DET}\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = 0.5$$ [M1,A1]

$$\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}^2 = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix}$$ [M1,A1]

$$\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}^{-1} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$$ [M1,A2]

$$\left[\mathrm{VECTOR}\left(\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}^n, n, [5, 10, 15, 20]\right)\right]$$

$$\left[\begin{bmatrix} 0.515625 & 0.484375 \\ 0.484375 & 0.515625 \end{bmatrix}, \begin{bmatrix} 0.500488 & 0.499511 \\ 0.499511 & 0.500488 \end{bmatrix}, \begin{bmatrix} 0.500015 & 0.499984 \\ 0.499984 & 0.500015 \end{bmatrix}, \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}\right]$$ [A3]

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$ [B2]

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\mathrm{DET}\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} - ¿\cdot \mathrm{IDENTITY\_MATRIX}(3)\right) = -¿^3 + 3\cdot¿^2 + 16\cdot¿ + 12$$ [M2,A2]

$$\mathrm{SOLVE}(-¿^3 + 3\cdot¿^2 + 16\cdot¿ + 12, ¿)$$

$$[¿ = -1, ¿ = -2, ¿ = 6]$$ [M1,A2]

(b)

$$\text{SOLVE}\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}\cdot[x,\ y,\ z] = -\ 1\cdot[x,\ y,\ z],\ [x,\ y,\ z]\right)$$

$$\left[\ x = @1 \quad y = -\ \frac{5\cdot @1}{2} \quad z = @1\ \right]$$

$$\text{SOLVE}\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}\cdot[x,\ y,\ z] = -\ 2\cdot[x,\ y,\ z],\ [x,\ y,\ z]\right)$$

$$\left[\ x = @2 \quad y = -\ \frac{3\cdot @2}{5} \quad z = -\ \frac{3\cdot @2}{5}\ \right]$$

$$\text{SOLVE}\left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}\cdot[x,\ y,\ z] = 6\cdot[x,\ y,\ z],\ [x,\ y,\ z]\right)$$

$$[\ x = @3 \quad y = @3 \quad z = @3\ ]$$

[M2,A4]

3

(a)

$z1 := 1 - i$

$z2 := 1 + \sqrt{2} \cdot i$

$z1 \cdot z2 = \sqrt{2} + 1 + i \cdot (\sqrt{2} - 1)$

$$\frac{z1}{z2} = -\frac{\sqrt{2}}{3} + \frac{1}{3} - i \cdot \left(\frac{\sqrt{2}}{3} + \frac{1}{3} \right)$$

$$\left| \frac{z1}{z2} \right| = \frac{\sqrt{6}}{3}$$

$z1 \cdot \mathrm{CONJ}(z1) = 2$

$$\mathrm{PHASE}(z1 \cdot z2) = \frac{\pi}{4} - \mathrm{ATAN}\left(\frac{\sqrt{2}}{2} \right)$$

$\mathrm{PHASE}(z1 \cdot z2) = 0.169918454727$

[A2 @ √]

(b) $e^{(2\pi i)} = 1$, so $e^{2\pi k i} = \left(e^{2\pi i}\right)^k = 1^k = 1$ [A1]

Hence $z = re^{i\theta}e^{2\pi i k} = re^{i(2\pi k + i\theta)}$ [A1]

but $r = |z|$ and $\theta = \arg z$ so [A1]

$z = |z|e^{i(2\pi k + \arg z)}$ hence $\ln z = \ln\left(|z|e^{i(2\pi k + \arg z)}\right) = \ln|z| + \ln e^{i(2\pi k + \arg z)}$ [A1]

$\ln z = \ln|z| + i(2\pi k + \arg z)$ [A1]

(c)

$(\cos(\theta) + i\sin(\theta))^3 = \cos(3\theta) + i\sin(3\theta)$ [A2]

$\mathrm{IM}\left((c + i \cdot s)^3\right) = s \cdot \left(3 \cdot c^2 - s^2\right)$ [A2]

$s \cdot \left(3 \cdot \sqrt{\left(1 - s^2\right)}^2 - s^2\right)$ [A2]

$s \cdot \left(3 - 4 \cdot s^2\right)$ [A2]

$\mathrm{SIN}(\theta) \cdot \left(3 - 4 \cdot \mathrm{SIN}(\theta)^2\right)$ [A2]

4.

(a) $f := \sin(x)\cdot\sin(y) + \sin(x\cdot y)$

$$\frac{d}{dx} f = y\cdot\cos(x\cdot y) + \cos(x)\cdot\sin(y)$$

$$\frac{d}{dy} f = x\cdot\cos(x\cdot y) + \sin(x)\cdot\cos(y)$$

$$\left(\frac{d}{dx}\right)^2 f = -y^2\cdot\sin(x\cdot y) - \sin(x)\cdot\sin(y)$$

$$\frac{d}{dy}\frac{d}{dx} f = \cos(x\cdot y) - x\cdot y\cdot\sin(x\cdot y) + \cos(x)\cdot\cos(y)$$

$$\frac{d}{dx}\frac{d}{dy} f = \cos(x\cdot y) - x\cdot y\cdot\sin(x\cdot y) + \cos(x)\cdot\cos(y)$$

[A2 @]

(b)

$$\frac{dz}{dt} = \frac{dx}{dt}\frac{\partial z}{\partial x} + \frac{dy}{dt}\frac{\partial z}{\partial y}$$ [A2]

$$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = \frac{1}{t}$$ [A2]

$$\frac{\partial z}{\partial x} = -2\cos x\sin x = -\sin(2x) \text{ and } \frac{\partial z}{\partial y} = -2\sin y\cos y = -\sin(2y)$$ [A2]

$$\frac{dz}{dt} = -2t\sin(2x) - \frac{2}{t}\sin(2y)$$

$$\frac{dz}{dt} = -2t\sin(2t^2) - \frac{2}{t}\sin(2\ln t)$$ [A2]

or

$$-\frac{2\cdot\sin(\ln(t))\cdot\cos(\ln(t))}{t} - 4\cdot t\cdot\sin(t^2)\cdot\cos(t^2)$$

(c)

$f := x^2 + y^2 + 2 \cdot x + 4 \cdot y - 3$

$\mathrm{SOLVE}\left(\left[\frac{d}{dx} f = 0, \frac{d}{dy} f = 0\right], [x, y]\right)$ [M1,A2]

$[\ x = -1 \quad y = -2\ ]$ [A1]

$\left(\frac{d}{dx}\right)^2 f = 2$ [A1]

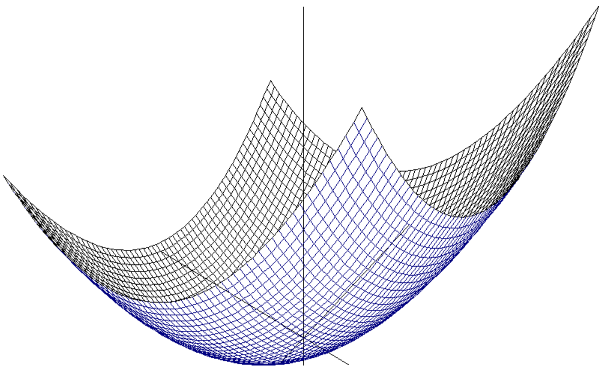
$\frac{d}{dy} \frac{d}{dx} f = 0$ [A1]

$\left(\frac{d}{dy}\right)^2 f = 2$

$d = -4$ [A1]

$a = 2$

therefore a local minimum at (-1,-2) [A2]



5 (a)

$$\frac{dy}{y^3} = \frac{dx}{x(x^2+1)}$$ [A2]

$$\int \frac{dy}{y^3} = \int \frac{dx}{x(x^2+1)} = \int \frac{1}{x} - \frac{x}{x^2+1}\,dx$$ [A2]

$$-\frac{y^{-2}}{2} = \ln x - \frac{\ln(x^2+1)}{2} + c$$ [A2]

$$y^{-2} = 2\ln\left(\frac{\sqrt{x^2+1}}{x}\right) + c \quad = \quad y^{-2} = \ln\left(\frac{(x^2+1)}{x^2}\right) + c$$ [A2]

$$c = 1 - \ln 2$$ [A2]

$$y^{-2} = \ln\left(\frac{(x^2+1)}{2x^2}\right) + 1$$ [A2]

(b)

Obviously not [A2]

$$(\ln y + x) + \frac{x}{y}\frac{dy}{dx} = 0$$ [A2]

$$p := LN(y) + x$$

$$q := \frac{x}{y}$$

$$\frac{d}{dy} p = \frac{1}{y}$$

$$\frac{d}{dx} q = \frac{1}{y}$$

an exact equation [A3]

$$\int p \, dx = x \cdot LN(y) + \frac{x^2}{2}$$

$$\int q \, dy = x \cdot LN(y)$$

$$x \cdot LN(y) + \frac{x^2}{2} = c$$

[M1,A3]

$$y = e^{\left(c/x - x/2\right)}$$

$$y := e^{c/x - x/2}$$

$$\frac{(LN(y) + x) \cdot y}{x} + \frac{d}{dx} y = 0$$ [A4]

(6) (a) (i) $m^2 - 3m + 2 = (m-1)(m-2) = 0$ [A4]

(ii) $y = Ae^{x} + Be^{2x}$ [A6]

(b)

$Y(x) := a\cdot SIN(x) + b\cdot COS(x)$ [M1,A2]

$$\left(\frac{d}{dx}\right)^2 Y(x) - 3\cdot\frac{d}{dx} Y(x) + 2\cdot Y(x)$$

$$(b - 3\cdot a)\cdot COS(x) + (a + 3\cdot b)\cdot SIN(x)$$

$$SOLVE([b - 3\cdot a = 0,\ a + 3\cdot b = 1],\ [a,\ b]) = \left[a = \frac{1}{10}\quad b = \frac{3}{10}\right]$$

$$\frac{1}{10}\cdot SIN(x) + \frac{3}{10}\cdot COS(x)$$ [M3,A3]

$$a\cdot e^{x} + b\cdot e^{2\cdot x} + \left(\frac{1}{10}\cdot SIN(x) + \frac{3}{10}\cdot COS(x)\right)$$ [A2]

$$Y(x) := a\cdot e^{x} + b\cdot e^{2\cdot x} + \left(\frac{1}{10}\cdot SIN(x) + \frac{3}{10}\cdot COS(x)\right)$$

$$\left(\frac{d}{dx}\right)^2 Y(x) - 3\cdot\frac{d}{dx} Y(x) + 2\cdot Y(x) = SIN(x)$$ [M1,A3]