5.

Solution

(i)

Till $p > FLOOR\left(\sqrt{n}\right)$ (1)

Divide by each prime number p, if remainder =0 then composite (1)
else Prime (1)

If composite, then n=r*s (at least) where r and s are integers
one has to $\leq \sqrt{n}$ and the other $\geq \sqrt{n}$. (2)

or

if we have not found a prime divisor before $\sqrt{n}$, then we cannot find a divisor after $\sqrt{n}$. If so the divisor p_1 would be greater that $\sqrt{n}$ and so would its composite pair, i.e. $p_2 = \dfrac{n}{p_1}$ (as p_2 would have been found before) (2) [5]

(ii)

```
FLOOR(√2371)=48
```
(1)

```
VECTOR(MOD(2371, p), p,
[3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47])

(or some method)
=[1, 1, 5, 6, 5, 8, 15, 2, 22, 15, 3, 34, 6, 21]
```
(1)

therefore prime (1)

```
VECTOR(MOD(2599, p), p,
 [3, 5, 7, 11, 13, 17, 19, 23])
```
(1) or some method

```
= [1, 4, 2, 3, 12, 15, 15, 0]
```
therefore not prime (1) [5]

(iii)

Express n in the form
$n = 1 + 2^k q$, where q is an odd number (1)
If $n = 1 + 2^k q$
and x a random integer less than n but not equal to 1, (1)
MOD($x^{2^{jq}}$,n) $= n - 1$ or 1 (1)
for some $0 \leq j \leq k$ -10<=j<=k-1, (1)
then n passes the Rabin Miller test. (1) [5]

(iv) 4235679031=1+2 2117839515 (1)
$k = 0$ only (1)

$$\text{MOD}\left(5^{2^{0}\cdot 2117839515},\ 4235679031\right) = 1$$ (2)

Therefore **probably** prime (1) [5]

(i) does not converge for $x \geq 1$ (1), oscillating signs (1), slow convergence if x close to 1 (1). [3]

(ii) Converges for all x (1), no oscillating signs (1), all even powers are 0 (1). [3]

(iii) Main features

```
LN(x):=
  x=(x-1)/(x+1)                              (1)
  xsq=x*x                                    }
  denom=3                                    } (2)
  numerat=3                                  }
  old=x
  new=x^3/3+x
  While( abs(old-new)<=(10^-10)abs(new),     }
    old=new                                  }
    numerat:* xsq                            } (3)
    denom:+2                                 }
    new:+numerat/denom                       }
  Else Return new                            (1)
```

[7]

(iv)

```
n := 0
Loop
  If a < 1
     RETURN [a, n]
  a :/ 10
  n :+ 1                                     (4)
```

If z is large, then transform z into $a \times 10^n$ and use the identity

$\ln z = \ln a + n \ln 10$ (3) [7]

7.

Solutions

(i) Must include Hessenberg form (1), QR factorization and RQ update(1), Francis shifts(1), quasi triangular matrix (1) and complex eigenvalue extraction (1). [1 mark each]

(ii)

$$\text{EIGENVALUES}\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = [5.791287847,\ 1.208712152]\ (1)$$

nearest eigenvalue to $A_{4,4} = 5.791287847$ (1)

Francis Shift

$= A - 5.791287847\text{I} =$

$$\begin{bmatrix} -3.791287846 & 2.1 & 1 & 3 \\ 3 & -3.791287846 & 2.3 & -1 \\ 0 & 0.3 & -3.791287846 & 3 \\ 0 & 0 & 1 & -0.7912878469 \end{bmatrix} \quad (2)$$

QR factorisation

$$Q = \begin{bmatrix} -0.7841903732 & -0.6107439506 & -0.1050073689 & 0.0317921019 \\ 0.6205203127 & -0.7718353723 & -0.1327043871 & 0.04017766962 \\ 0 & 0.1768105901 & -0.9420168514 & 0.2852056568 \\ 0 & 0 & 0.2897710296 & 0.9570959985 \end{bmatrix}$$

$$\mathbf{R} := \begin{bmatrix} 4.834652369 & -3.999370903 & 0.6430063459 & -2.973091432 \\ 0 & 1.696730946 & -3.056305148 & -0.5299647092 \\ 0 & 0 & 3.45100061 & -3.237660568 \\ 0 & 0 & 0 & 0.1534771744 \end{bmatrix}$$

$$\mathbf{R \cdot Q} = \begin{bmatrix} -6.272978728 & 0.2478115737 & -1.444178639 & -2.669126507 \\ 1.052856017 & -1.849984078 & 2.500358892 & -1.310731924 \\ 0 & 0.6101734542 & -4.189080965 & -2.114507078 \\ 0 & 0 & 0.04447323884 & 0.1468923894 \end{bmatrix} \quad (2)$$

(iii)

next eigenvalues [-4.167283294, 0.1250947193]

Francis shift=0.1250947193 (2)

$$\begin{bmatrix} -6.272978728 & 0.2478115737 & -1.444178639 & -2.669126507 \end{bmatrix}$$

$$\begin{bmatrix} 1.052856017 & -1.849984078 & 2.500358892 & -1.310731924 \\ 0 & 0.6101734542 & -4.189080965 & -2.114507078 \\ 0 & 0 & 0.04447323884 & 0.1468923894 \end{bmatrix} - 0.1250947193 \cdot I$$

$$\begin{bmatrix} -6.398073447 & 0.2478115736 & -1.444178639 & -2.669126506 \\ 1.052856017 & -1.975078797 & 2.500358892 & -1.310731924 \\ 0 & 0.6101734542 & -4.314175684 & -2.114507077 \\ 0 & 0 & 0.04447323883 & 0.02179767010 \end{bmatrix} \quad (2)$$

RQ application

$$\begin{bmatrix} -6.489851603 & 0.08594169960 & -1.863273215 & -2.445242439 \\ 0.3253645205 & -2.930921587 & 2.687647658 & -0.9660744142 \\ 0 & 1.044378913 & -3.233618867 & -2.582111748 \\ 0 & 0 & 0.0001444418225 & -0.01113819759 \end{bmatrix} \quad (2)$$

approx eigenvalue = −0.01113819759 +shift values (1)

=−0.01113819759 + 0.1250947193 + 5.791287847=5.905244368 (1)

(actual eigenvalues= 5.905057141)

8. (i)

Orthogonal (1)
Inverse=transpose (1)
Annihilates elements of matrices (1)

(ii)

$$v1 := \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$u1 := v1 + \begin{bmatrix} \text{SIGN}(v1_{1,1}) \cdot |v1| \\ 0 \\ 0 \end{bmatrix}$$ (1)

$$\text{IDENTITY\_MATRIX}(3) - \frac{2 \cdot (u1 \cdot u1')}{(u1' \cdot u1)_{1,1}}$$ (2)

$$H1 := \begin{bmatrix} -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} & \frac{2 \cdot \sqrt{6}}{15} + \frac{1}{5} & \frac{\sqrt{6}}{15} - \frac{2}{5} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{15} - \frac{2}{5} & \frac{\sqrt{6}}{30} + \frac{4}{5} \end{bmatrix}$$ (1)

or

$$H1 := \begin{bmatrix} -0.4082482904 & -0.8164965809 & -0.4082482904 \\ -0.8164965809 & 0.5265986323 & -0.2367006838 \\ -0.4082482904 & -0.2367006838 & 0.8816496580 \end{bmatrix}$$

(iii)

$$A := \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

H1·A=

$$\begin{bmatrix} -\sqrt{6} & -\frac{5 \cdot \sqrt{6}}{6} & -\frac{5 \cdot \sqrt{6}}{6} \\ & \sqrt{6} \quad 3 & 7 \cdot \sqrt{6} \quad 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\frac{}{15}-\frac{}{5} & -\frac{}{15}-\frac{}{5} \\ 0 & \frac{6}{5}-\frac{\sqrt{6}}{30} & \frac{2}{5}-\frac{7\cdot\sqrt{6}}{30} \end{bmatrix} \qquad \textbf{(1)}$$

$$v2 := \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{15}-\frac{3}{5} \\ \frac{6}{5}-\frac{\sqrt{6}}{30} \end{bmatrix}$$

$$u2 := v2 + \begin{bmatrix} 0 \\ SIGN(v1_{1,1})\cdot\|v1\| \\ 0 \end{bmatrix} \qquad (1)$$

$$IDENTITY\_MATRIX(3) - \frac{2\cdot(u2\cdot u2')}{(u2'\cdot u2)_{1,1}} \qquad (2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2112\cdot\sqrt{6}}{36745}-\frac{19467}{36745} & -\frac{12474\cdot\sqrt{6}}{36745}-\frac{3296}{36745} \\ 0 & -\frac{12474\cdot\sqrt{6}}{36745}-\frac{3296}{36745} & \frac{19467}{36745}-\frac{2112\cdot\sqrt{6}}{36745} \end{bmatrix}$$

$$H2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3\cdot\sqrt{66}}{55}-\frac{2\cdot\sqrt{11}}{55} & \frac{6\cdot\sqrt{66}}{55}-\frac{\sqrt{11}}{55} \\ 0 & \frac{6\cdot\sqrt{66}}{55}-\frac{\sqrt{11}}{55} & \frac{3\cdot\sqrt{66}}{55}+\frac{2\cdot\sqrt{11}}{55} \end{bmatrix} \qquad (1)$$

OR

$$H2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5637339053 & 0.8259564661 \\ 0 & 0.8259564661 & 0.5637339053 \end{bmatrix}$$

$$R = H2\cdot H1\cdot A \qquad (2)$$

$$R := \begin{bmatrix} -\sqrt{6} & -\frac{5\cdot\sqrt{6}}{6} & -\frac{5\cdot\sqrt{6}}{6} \\ 0 & \frac{\sqrt{66}}{6} & \frac{5\cdot\sqrt{66}}{66} \end{bmatrix} \qquad (1)$$

$$\begin{bmatrix} 0 & 0 & -\frac{4\cdot\sqrt{11}}{11} \end{bmatrix}$$

OR

$$R := \begin{bmatrix} -2.449489742 & -2.041241452 & -2.041241452 \\ 0 & 1.354006400 & 0.6154574548 \\ 0 & 0 & -1.206045378 \end{bmatrix}$$

(iv)

Q = H1\`·H2\` (3)

$$Q := \begin{bmatrix} -\frac{\sqrt{6}}{6} & \frac{\sqrt{66}}{66} & -\frac{3\cdot\sqrt{11}}{11} \\ -\frac{\sqrt{6}}{3} & -\frac{2\cdot\sqrt{66}}{33} & \frac{\sqrt{11}}{11} \\ -\frac{\sqrt{6}}{6} & \frac{7\cdot\sqrt{66}}{66} & \frac{\sqrt{11}}{11} \end{bmatrix} \quad (1)$$

OR

$$Q := \begin{bmatrix} -0.4082482904 & 0.1230914909 & -0.9045340337 \\ -0.8164965809 & -0.4923659639 & 0.3015113445 \\ -0.4082482904 & 0.8616404368 & 0.3015113445 \end{bmatrix}$$

(v)

$$Q\cdot R = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad (1)$$