1.

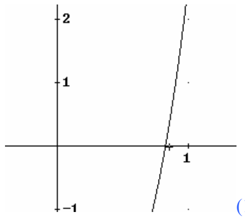
(i) draw tangent at $(x_0, f(x_0))$ (1)

x_1 is the intersection of the tangent with x axis (1)

develops $x - \dfrac{f}{\frac{d}{dx} f}$ (2)

near a turning point or near an asymptote or steep gradient (1) [5]

(ii) $f(x) = \sqrt{x}\, e^{2x} - \sin x - 4$ (1)



(1)

$f(0) = -4,\ f(1) = 2.55$ (1)

so a solution in [0,1] (1) [4]

(iii)

$$x - \frac{\sqrt{x}\cdot ê^{2\cdot x} - \text{SIN}(x) - 4}{\frac{d}{dx}\left(\sqrt{x}\cdot ê^{2\cdot x} - \text{SIN}(x) - 4\right)}$$ (2)

$$x - \frac{2\cdot\sqrt{x}\cdot\left(\sqrt{x}\cdot ê^{2\cdot x} - \text{SIN}(x) - 4\right)}{ê^{2\cdot x}\cdot(4\cdot x + 1) - 2\cdot\sqrt{x}\cdot \text{COS}(x)}$$ (1) [3]

(iv)

$$\text{ITERATES}\left(x - \frac{\sqrt{x}\cdot ê^{2\cdot x} - \text{SIN}(x) - 4}{\frac{d}{dx}\left(\sqrt{x}\cdot ê^{2\cdot x} - \text{SIN}(x) - 4\right)},\ x,\ 1,\ 5\right)$$ (1)

[1, 0.8579334650, 0.8267239644, 0.8254374665, 0.8254353751,
0.8254353751] (1) [2]

(v) $x = 0.82544$ (1)

$f(0.825435) = -4.374532927\cdot 10^{-6}$ (1)

$f(0.825445) = 0.0001122221933$ (1)

sign change so correct to 6 sig figs. [3]

(vi) Gradient infinite at $x = 0$, (2) hence $x_0 = x_1 = x_2 = \cdots$ (1) [3]

2 (i) Pick a starting vector $\mathbf{v_0}$ (1)

evaluate $A\mathbf{v}_0$ (1)

normalise $\|A\mathbf{v}_0\|$ (1)

iterate $\mathbf{v_{n+1}} = \|A\mathbf{v_n}\|$ (1)

until $|\mathbf{v_{n+1}} - \mathbf{v_n}| < \varepsilon$ (1) [5]

(ii)

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \cdot [1, 1, 1]$$

[3, 9, 11] (1)

$$\left\| \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \cdot [1, 1, 1] \right\|$$

$\sqrt{211}$ (1)

$$\frac{[3, 9, 11]}{\sqrt{211}}$$

[0.2065285172, 0.6195855517, 0.7572712299] (1) [3]

(iii)

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \cdot [0.2065285172, 0.6195855517, 0.7572712299]$$

[1.308013942, 5.300898609, 6.677755390]

$$\frac{[1.308013942, 5.300898609, 6.67775539]}{|[1.308013942, 5.300898609, 6.67775539]|}$$

[0.1516413374, 0.6145464729, 0.7741689333]

(2)

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \cdot [0.1516413374, 0.6145464729, 0.7741689333]$$

[1.221111822, 5.243597826, 6.632313232]

$$\frac{[1.221111822, 5.243597826, 6.632313232]}{|[1.221111822, 5.243597826, 6.632313232]|}$$

[0.1429457760, 0.6138259797, 0.7763917643]

(2)

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \cdot [0.142945776, 0.6138259797, 0.7763917643]$$

[1.208031950, 5.232936548, 6.623154292]

$$\frac{[1.20803195, 5.232936548, 6.623154292]}{|[1.20803195, 5.232936548, 6.623154292]|}$$

$$[0.1416719657,\ 0.6136927153,\ 0.7767305229]$$ (2)

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \cdot [0.1416719657,\ 0.6136927153,\ 0.7767305229]$$

$$[1.206019588,\ 5.231344168,\ 6.621767407]$$

$$\frac{[1.206019588,\ 5.231344168,\ 6.621767407]}{\left|[1.206019588,\ 5.231344168,\ 6.621767407]\right|}$$

$$[0.1414747798,\ 0.6136743313,\ 0.7767809869]$$

$$[0.141,\ 0.614,\ 0.777]$$ (2)

[8]

(iv)

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \cdot [0.1414747798,\ 0.6136743313,\ 0.7767809869] = \lambda \cdot [0.1414747798,\ 0.6136743313,\ 0.7767809869]$$

$$[1.205716786,\ 5.231096501,\ 6.621551819] = \lambda \cdot [0.1414747798,\ 0.6136743313,\ 0.7767809869]$$ (2)

$$\lambda = \frac{1.205716786}{0.1414747798} = 8.522485687$$ (2) (or using any of the other elements) [4]

3(a) (i) Evaluate gradient at (x_0, y_0), i.e. k_1. (1)

Use normal Euler method to get a first estimate of y_1; (1)

use this estimate of y_1 and x_1 to approx the grad at (x_1, y_1) i.e. k_2. (1)

Calculate $\frac{k_1 + k_2}{2}$ average. (1)

Use this average in the normal Euler method (1) [5]

(ii)

$f(x, y) := e^{-x} - x \cdot y^2$

k1 := f(0, 1)

k1 = 1 (1)

k2 := f(0.1, 1 + 0.1·k1) (2)

k2

0.7838374180 (1)

$1 + \frac{0.1}{2} \cdot (k1 + k2) = 1.089191870$ (1) [5]

(b)

f(x, y, t) := x + y − 2·t

$g(x, y, t) := x \cdot y^2 + t$

k1 := f(1, 1, 0)

k1 = 2 (1)

l1 := g(1, 1, 0)

l1 = 1 (1)

k2 := f(1 + 0.1·k1, 1 + 0.1·l1, 0.1) (1)

k2 = 2.1 (1)

l2 := g(1 + 0.1·k1, 1 + 0.1·l1, 0.1) (1)

l2 = 1.552 (1)

$x1 := 1 + \frac{0.1}{2} \cdot (k1 + k2)$ (1)

x1 = 1.205 (1)

$y1 := 1 + \frac{0.1}{2} \cdot (l1 + l2)$ (1)

y1 = 1.1276 (1) [10]

4 (a) (i) If **u** and **v** belong to W, then:
u+**v** is in W (1)
and k**u** (or k**v**) is in W (1) [2]

(ii) Let $\mathbf{u} = \begin{pmatrix} 0 & a_1 \\ b_1 & 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 & a_2 \\ b_2 & 0 \end{pmatrix}$, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$ (1)

then $\mathbf{u} + \mathbf{v} = \begin{pmatrix} 0 & a_1 \\ b_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_2 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 + a_2 \\ b_1 + b_2 & 0 \end{pmatrix}$ which is in W (1)

and $k\mathbf{u} = \begin{pmatrix} 0 & ka_1 \\ kb_1 & 0 \end{pmatrix}$ is also in W (1). [3]

(b) $k_1\mathbf{v_1} + k_2\mathbf{v_2} + k_3\mathbf{v_3} = (x, y, z) \; \forall \; x, y, z \in \mathbb{R}$ if they are to span the space. (2)

$$\begin{aligned} k_1 + k_2 + 2k_3 &= x \\ \therefore \quad k_1 + 0 + k_3 &= y \qquad (3) \\ 2k_1 + k_2 + 3k_3 &= z \end{aligned}$$

this system of equations will only have solutions for k_1, k_2 and k_3 if

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} \neq 0$$

however,

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 0 \quad (1)$$ [6]

(c) (i) The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly **dependent** set if
$k_1\mathbf{v_1} + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ (1)
has solutions other than $k_1 = k_2 = k_3 = 0$ (1) [2]

(ii)
$$\begin{pmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ 1 & 5 & 1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (2)$$

the square sub matrix

$\begin{pmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ 1 & 5 & 1 \end{pmatrix}$ is NOT invertible as $\begin{vmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ 1 & 5 & 1 \end{vmatrix} = 0$ (1)

hence there are infinite solutions to the linear set

$$\begin{pmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ 1 & 5 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

solving using Derive we get

$k_1 = \lambda, k_2 = -\frac{2}{7}\lambda, k_3 = \frac{3\lambda}{7}$ (2)

using fourth row of matrix

$-k_1 + k_2 + 3k_1 = -\lambda + \frac{-2\lambda}{7} + 3\frac{3\lambda}{7} = 0$ (1)

Hence $k_1 = \lambda, k_2 = -\frac{2}{7}\lambda, k_3 = \frac{3\lambda}{7}$ is consistent for all four equations, so the vectors are linearly dependent. (1) [7]

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5.(i)

$$F(x) := \frac{\text{SIN}(x)}{1 + x^2}$$

```
VECTOR([n, M(0, 1, n)], n, [2, 4, 8, 16])
```

$$\begin{bmatrix} 2 & 0.3345497957 \\ 4 & 0.3248328282 \\ 8 & 0.3225452522 \\ 16 & 0.3219809778 \end{bmatrix}$$

(6)

[6]

(ii) order of midpoint is h^2

$I = M_n + ch^2 + \cdots$ (1)

$I = M_{2n} + c\left(\frac{h}{2}\right)^2 + \cdots$ (1)

$\rightarrow I = \frac{4M_{2n} - M_n}{3} + \cdots$ (1)

$$R1(v) := \text{VECTOR}\left(\frac{4 \cdot v_{r+1} - v_r}{3}, r, 1, \text{DIM}(v) - 1\right)$$

```
R1([0.3345497957, 0.3248328282, 0.3225452522,0.3219809778])
[0.3215938390, 0.3217827268, 0.3217928863]
```

(2) [5]

(iii) h^4 (order 4) [2]

(iv) Extrapolating order 4 we get $I = \frac{(2^4 R_{2,2n} - R_{2,n})}{2^4 - 1} + \cdots$ (2)

$$R2(v) := \text{VECTOR}\left(\frac{2^4 \cdot v_{r+1} - v_r}{2^4 - 1}, r, 1, \text{DIM}(v) - 1\right)$$

```
R2([0.3215938389, 0.3217827267, 0.3217928862])
[0.3217953192, 0.3217935635]
```

(2)

Extrapolating order 6 we get $I = \dfrac{(2^6 R_{4,2n} - R_{4,n})}{2^6 - 1} + \cdots$ (2)

```
R3(v) := VECTOR((2^6·v_(r+1) - v_r)/(2^6 - 1), r, 1, DIM(v) - 1)

R3([0.3217953192, 0.3217935635])

[0.3217935356]
```

(1) [7]

6.

(i) Testing gradients (1) that over estimate (1) and under estimate the target (1), use of linear interpolation (1)or systematic search (1). [5]

(ii)
$$\frac{dy}{dx} = z \quad (1)$$
$$\frac{dz}{dx} - xz + 4y = 5e^{-2x}\sin x \quad (2)$$

$$\frac{dz}{dx} = xz - 4y + 5e^{-2x}\sin x \quad (1)$$
[4]

(iii)

```
(RK([z, x·z - 4·y + 5·e^(- 2·x)·SIN(x)], [x, y, z], [0, 0, 1], 0.05, 20))_-1
[1, 0.8164230289, 0.1567178803]
(RK([z, x·z - 4·y + 5·e^(- 2·x)·SIN(x)], [x, y, z], [0, 0, -1], 0.05, 20))_-1
[1, -0.2348193848, 0.8979081400]
(RK([z, x·z - 4·y + 5·e^(- 2·x)·SIN(x)], [x, y, z], [0, 0, -0.8], 0.05, 20))_-1
[1, -0.1296951434, 0.8237891140]
(RK([z, x·z - 4·y + 5·e^(- 2·x)·SIN(x)], [x, y, z], [0, 0, -0.5], 0.05, 20))_-1
[1, 0.02799121859, 0.7126105751]
(RK([z, x·z - 4·y + 5·e^(- 2·x)·SIN(x)], [x, y, z], [0, 0, -0.55], 0.05, 20))_-1
[1, 0.001710158251, 0.7311403316]
```

(8) marks for a value of dy/dx that provides a solution within the specified error bounds

7.

(i) $E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ [2]

(ii) $E_1^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ [1]

(iii) $E_1 A = \begin{pmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow E_2 E_1 A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{pmatrix}$ (2)

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \rightarrow E_3 E_2 E_1 A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{pmatrix}$ (2)

$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix} \rightarrow E_4 E_3 E_2 E_1 A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{pmatrix}$ (2)

$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix} \rightarrow E_5 E_4 E_3 E_2 E_1 A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ (2) [8]

(iv) $E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}, E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}, E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$= \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}$ (3)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$ [4]

(v) $$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \to y_1 = 1$$

$$-3y_1 + y_2 = 2 \to y_2 = 5$$

$$4y_1 - 3y_2 + 7y_3 \to y_3 = 2 \quad (3)$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \to x_3 = 2$$

$$x_2 + 3x_3 = 5 \to x_2 = -1 \quad (2)$$ [5]

$$x_1 + 3x_2 + x_3 = 1 \to x_1 = 2$$

8 (i)

In row echelon form

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{vmatrix} = (1)(-\overset{(1)}{3})(-2) = 6 \text{ (1)}$$

[2]

(ii)

Interchange R1 and R2

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \text{ (2)}$$

Add -2/3R1 to R3

[-2,4,-6]+[2,6,1] → [0, 10,-5]

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \text{ (3)}$$

Add -10R2 to R3

[0,-10,-50]+[0,10,-5] → [0,0,-55]

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \text{ (3)}$$

[8]

(iii)

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -3(1)(-55) = 165$$

[3]

(iv)

$$\begin{vmatrix} 0 & 2 & 10 \\ -3 & 6 & -9 \\ 6 & 18 & 3 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 & 5 \\ -3 & 6 & -9 \\ 6 & 18 & 3 \end{vmatrix} = 2(-1)\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 6 & 18 & 3 \end{vmatrix} = (2)(-1)(3)\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} \text{ (3)}$$

=-6 x 165=-990 (1)

[4]

(v)

As DET(AB)=DET(A)DET(B)

$$\left|\begin{pmatrix} 0 & 2 & 10 \\ -3 & 6 & -9 \\ 6 & 18 & 3 \end{pmatrix}\begin{pmatrix} 0 & -3 & 6 \\ 2 & 6 & 18 \\ 10 & -9 & 13 \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & 2 & 10 \\ -3 & 6 & -9 \\ 6 & 18 & 3 \end{pmatrix}\right|\left|\begin{pmatrix} 0 & -3 & 6 \\ 2 & 6 & 18 \\ 10 & -9 & 13 \end{pmatrix}\right| \text{ (1)}$$

As DET(A`)=DET(A)

$$\left|\begin{pmatrix} 0 & 2 & 10 \\ -3 & 6 & -9 \\ 6 & 18 & 3 \end{pmatrix}\right|\left|\begin{pmatrix} 0 & -3 & 6 \\ 2 & 6 & 18 \\ 10 & -9 & 13 \end{pmatrix}\right| = \left|\begin{pmatrix} 0 & 2 & 10 \\ -3 & 6 & -9 \\ 6 & 18 & 3 \end{pmatrix}\right|\left|\begin{pmatrix} 0 & 2 & 10 \\ -3 & 6 & -9 \\ 6 & 18 & 3 \end{pmatrix}\right| \text{ (1)}$$

$$= (-990)^2 = 980100 \text{ (1)}$$

[3]
Total [20]