Liverpool John Moores University
SCHOOL OF COMPUTING AND MATHEMATICAL SCIENCES

Semester 2 Examinations 2000

CMSMA3016 Computational Mathematics

Mark Scheme

Question 1

(a)
IEEE standard precision format using a 32 bit word.
• The left-most bit is used for the sign of the number, 0 = positive and 1 = negative. (1)
• The next 8 bits are reserved for the negative of the exponent of 2 needed to normalize the number (i.e. put it in the form .1xxxx.. , base 2). (2)
• The exponent is stored in excess 127 format. (1)
• The remaining 23 bits are used for the number itself in normalized form.(1)
· The leading one is not included since all normalized numbers begin with a 1.(2)

[B7]

(b)
•0.625 (decimal) = 0.101 (binary) Note that this number is already normalized so the exponent in excess 128 is:
· 10000000
• The sign is 0 since it is positive. Thus the representation is:
· 0| 100000000| 010 . . . 0 where the 0 . . . 0 stands for 21 0's

[B3]

(c)
• To multiply .625 by 2.5 we first put 2.5 = 10.1(binary) into IEEE floating point format, note that its exponent will be 2, or 10(binary).
• To multiply the two numbers we first write the numbers, restoring the implied 1 in scratch registers as integers by shifting the number the appropriate number of spaces to the left determined by the sum of the exponents. One number is chosen as the multiplicand and one as the multiplier.
•The product is achieved by shifting the multiplicand according to the bits of the multiplier and adding this to the result.
•When the result has been calculated, it is renormalized.
• The exponent is the sum of the exponents of the two numbers adjusted plus the exponent of two needed to normalize the product.

· [B5]

(d)

- Two's compliment arithmetic can be done within the machinery using normal binary addition hardware without resorting to any additional algorithms to do the arithmetic. (2)
- If signed magnitude format is used, an algorithm for subtraction that would implement borrows would need to be included in the machine code.
- With two's compliment a number is negated by flipping the bits of the original number and incrementing.
- These operations are easily included in the circuitry.

. [B5]

(e)

There is no positive number to corresspond to the two's compliment of 100000 . . . 0. It is its own complement.

[B3]

(f)

This is impossible since one can only have an even number of patterns in an *n*-bit binary word. Since there is only one zero.

[B2]

Question 2

(a) start with a=2

$$MOD(2^{4294967295-1}, 4294967295) = 1073741824$$

So we can definitely say that 4294967295 is not prime

[M2,A1]

(b)

$$MOD(2^{4294967297-1}, 4294967297) = 1$$

we may inadvisable use Fermat's little theorem to deduce that 4294967297 is prime, but it is necessary condition for *p* to be prime not sufficient

[B3]

$$MOD(3^{4294967297-1}, 4294967297) = 3029026160$$

so we have proven that 4294967297 is actually not prime!!

[M1,A1]

(c)

- All the numbers in column 2 are multiples of 2 (so after the first 2 every number is composite)
- All the numbers in column 3 are multiples of 3 (so after the first 3 every number is composite)
- All the numbers in column 4 are multiples of 2 (so every number is composite)
- All the numbers in column 6 are multiples of 6 (so every number is composite)
- So all the primes must be in column 1 and column 5
- Starting with column 5 we traverse columns 1 and 5 by adding 2, 4, 2, 4, ...

. [B5]

(d)

- Generate a list L that includes all the prime numbers up to √n (1)
- √n need be considered because if there is no prime factor <√n then there can no factor pair >√n to complement it. So it is prime(3).
- Systematically divided n by each member of L, if there always a remainder then n is prime(2).

. [B6]

(e)

```
FLOOR(√1361) = 36

L := [2, 3, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35]

VECTOR(MOD(1361, s), s, L) = [1, 2, 1, 3, 8, 9, 1, 12, 4, 11, 27, 28, 31]
```

no 0 mod so 1361 is prime

[M3,A3]

(f)

```
VECTOR(MOD(1357, s), s, L) = [1, 1, 2, 6, 4, 5, 14, 8, 0, 7, 23, 24, 27]
```

there is a 0 mod so 1361 is not prime (/ by 23) [M3,A3]

Question 3

(a)

- Only convergent for |x|<1
- Slowly convergent for |x|<1 beacuse general term is (-1)^(n+1) x^n /n

. [B2]

(b)

$$\mathrm{LN}\left(\frac{1+x}{1-x}\right)$$

$$\mathrm{TAYLOR}\left(\mathrm{LN}\left(\frac{1+x}{1-x}\right), x, 0, 10\right)$$

$$\frac{2\cdot x^9}{9} + \frac{2\cdot x^7}{7} + \frac{2\cdot x^5}{5} + \frac{2\cdot x^3}{3} + 2\cdot x$$

[B2]

- powers increase by 2
- z=(1+x)/(1-x), if 0<z<1 then 0<x<1/3 . So will converge quicker than ln(1+x)

. [B2]

(c)

find the value of n for which the term we add on is first <10^-6

$$\mathrm{SOLVE}\left(\frac{2\cdot\left(\frac{1}{3}\right)^n}{n} = 10^{-6}, n\right)$$

$$n = 11.02188213$$

$$\mathrm{TAYLOR}\left(\mathrm{LN}\left(\frac{1+x}{1-x}\right), x, 0, 11\right)$$

$$\frac{2\cdot x^{11}}{11} + \frac{2\cdot x^9}{9} + \frac{2\cdot x^7}{7} + \frac{2\cdot x^5}{5} + \frac{2\cdot x^3}{3} + 2\cdot x$$

$$\frac{2\cdot\left(\frac{1}{3}\right)^{11}}{11} + \frac{2\cdot\left(\frac{1}{3}\right)^{9}}{9} + \frac{2\cdot\left(\frac{1}{3}\right)^{7}}{7} + \frac{2\cdot\left(\frac{1}{3}\right)^{5}}{5} + \frac{2\cdot\left(\frac{1}{3}\right)^{3}}{3} + 2\cdot\frac{1}{3}$$

0.6931470737

so to 6 sig figures ln(2)=0.693147 [B2]

(d)

$LN(z) = LN(2^p \cdot x) = p \cdot LN(2) + LN(x)$

where $0.5 < x \le 1$, so if we find p and x we can find ln(z)

$$\frac{1000}{1024}$$

0.9765625

$z = 2^{10} \cdot 0.9765625$

we already know that ln(2)=0.693147

$LN(1000) = 10 \cdot LN(2) + LN(0.9765625)$

[M2,A1]

$$\frac{0.9765625 - 1}{0.9765625 + 1}$$

−0.0118577075

$$\frac{2 \cdot (-0.0118577075)^{11}}{11} + \frac{2 \cdot (-0.0118577075)^{9}}{9} + \frac{2 \cdot (-0.0118577075)^{7}}{7} + \frac{2 \cdot (-0.0118577075)^{5}}{5} + \frac{2 \cdot (-0.0118577075)^{3}}{3} + 2 \cdot (-0.0118577075)$$

−0.02371652661

−0.023717

$10 \cdot 0.693147 - 0.023717$

6.907752999

6.90775

[M1,A1]

(e)

$y = LN(\sqrt{(x^2 + 1)} + x)$

[M1,A1]

(f)

$y = LN(\sqrt{(2^2 + 1)} + 2)$

$y = LN(4.23606)$

$y = 1.44363$

[M1,A1]

(e)

$$\sqrt{(x^2 + 1)} + x$$

$$\sqrt{(1000^2 + 1)} + 1000$$

2000.0005

$$\frac{2000.0005}{2048}$$

0.9765627441

$$11 \cdot LN(2) + LN(0.9765627441)$$

$$\frac{0.9765627441 - 1}{0.9765627441 + 1}$$

−0.01185758254

$$\frac{2\cdot(-0.01185758254)^{11}}{11} + \frac{2\cdot(-0.01185758254)^{9}}{9} + \frac{2\cdot(-0.01185758254)^{7}}{7} + \frac{2\cdot(-0.01185758254)^{5}}{5} + \frac{2\cdot(-0.01185758254)^{3}}{3} + 2\cdot(-0.01185758254)$$

−0.02371627664

$$11 \cdot 0.693147 - 0.02371627664$$

7.600900723

7.60090 so 6 sig. figs

[M2,A2]

Question 4

(a)

$$T5(x) := 16 \cdot x^5 - 20 \cdot x^3 + 5 \cdot x$$

[M2,A2]

(b)

TAYLOR(SIN(x), x, 0, 5)

$$\frac{x^5}{120} - \frac{x^3}{6} + x$$

$$\frac{x^5}{120} - \frac{x^3}{6} + x - \frac{1}{120} \cdot \frac{1}{16} \cdot T5(x)$$

$$\frac{x \cdot (383 - 60 \cdot x^2)}{384}$$

[M2,A3]

(c)

$$\text{VECTOR}\left(\left[\text{SIN}(x) - \left(\frac{x^5}{120} - \frac{x^3}{6} + x\right), \text{SIN}(x) - \frac{x \cdot (383 - 60 \cdot x^2)}{384}\right], x, [0.2, 0.4, 0.6, 0.8]\right)$$

$$\begin{bmatrix} -2.538593242 \cdot 10^{-9} & 0.000440164128 \\ -3.243581509 \cdot 10^{-7} & 0.0004600089751 \\ -5.526605384 \cdot 10^{-6} & -4.502660538 \cdot 10^{-5} \\ -4.124243447 \cdot 10^{-5} & -0.0005605757678 \end{bmatrix}$$

[A4]

(d)

- less terms to compute for same accuracy away from point of expansion. (1)
- error is distributed over the interval of interest. (2)

[B3]

(e)

$$\mathrm{TAYLOR}(\mathrm{SIN}(x),\ x,\ 0,\ 5) - \frac{a + b\cdot x + c\cdot x^2 + d\cdot x^3}{1 + e\cdot x + f\cdot x^2} = 0$$

$$- \frac{x\cdot(b\cdot f^2 - c\cdot e\cdot f + d\cdot(e^2 - f)) + a\cdot f^2 - c\cdot f + d\cdot e}{f^2\cdot(f\cdot x^2 + e\cdot x + 1)} + \frac{x^5}{120} - \frac{x^3}{6} + \frac{x\cdot(f - d)}{f} - \frac{c\cdot f - d\cdot e}{f^2} = 0$$

$$\frac{f\cdot x^7 + e\cdot x^6 + x^5\cdot(1 - 20\cdot f) - 20\cdot e\cdot x^4 - 20\cdot x^3\cdot(6\cdot d - 6\cdot f + 1) + 120\cdot x^2\cdot(e - c) + 120\cdot x\cdot(1 - b) - 120\cdot a}{120\cdot(f\cdot x^2 + e\cdot x + 1)} = 0$$

equate co-efficients of x and solve

SOLUTIONS([a = 0, 1 − b = 0, e − c = 0, 6·d − 6·f + 1, e = 0, 1 − 20·f − 0], [a, b, c, d, e, f])

$$\left[\left[0,\ 1,\ 0,\ -\frac{7}{60},\ 0,\ \frac{1}{20}\right]\right]$$

[M3,A3]

$$\frac{a + b\cdot x + c\cdot x^2 + d\cdot x^3}{1 + e\cdot x + f\cdot x^2}$$

$$\frac{0 + 1\cdot x + 0\cdot x^2 + \left(-\frac{7}{60}\right)\cdot x^3}{1 + 0\cdot x + \frac{1}{20}\cdot x^2}$$

$$\frac{x\cdot(60 - 7\cdot x^2)}{3\cdot(x^2 + 20)}$$

$$\frac{0.4\cdot(60 - 7\cdot 0.4^2)}{3\cdot(0.4^2 + 20)}$$

0.3894179894

[A3]

Question 5

(a)

The range of A is a space of dimension at most __3__ located within a space of dimension __5__.

[B2]

(b)

```
[-0.179605302, -0.8980265101, 0.179605302, 0, 0.359210604]
```

in approximate form

```
u1 := SIGN(A COL 1)

u2 := SIGN(A COL 2 - (u1 • A COL 2)·u1)
```

[2]

```
[0.9488108006, -0.0777713771, 0.1762817881, -0.1607275126, 0.1918360635]

u3 := SIGN(A COL 3 - (u1 • A COL 3)·u1 - (u2 • A COL 3)·u2)
```

[3]

```
[0.06733188018, 0.1620952671, 0.1147135736, 0.9002521758, 0.381547321]

u1 := SIGN(A COL 1)

R := [u1, u2, u3]·A
```

[3]

So an orthonormal basis for the space is
[-0.170605, -0.898026, 0.179605, 0, 0.359210] ,
[0.948810, -0.0777713, 0.176281, -0.160727, 0.191836], and
[0.0673318, 0.162095, 0.114713, 0.900252, 0.381547]. T
his means that the space is, indeed three dimensional.

(c)

$$\begin{bmatrix} 5.567764362 & -0.538815906 & -0.359210604 \\ 2.034064217\cdot 10^{-12} & 6.221710168 & -4.852933931 \\ 3.861234041\cdot 10^{-13} & -1.673104493\cdot 10^{-13} & 8.019975062 \end{bmatrix}$$

$$A := \begin{bmatrix} -1 & 6 & -4 \\ -5 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & -1 & 8 \\ 2 & 1 & 2 \end{bmatrix}$$

Question 6

(a)

- Convert matrix to upper Hessngberg form using Householder reflections.
- * Find the eigenvalues of the bottom right 2x2 sub matrix, subtract (Francis shift) the diagonal elements by the nearest eigenvalue to the bottom diagonal element.
- QR factorise the matrix, evaluate RQ.
- If the bottom sub diagonal element is less than some pre-defined tolerance close to 0; then one eigenvalue is the bottom diagonal + sum of all the shifts.
- If the bottom sub diagonal element converges to a number other than 0; then two eigenvalues = sum of shifts +eigenvalues(bottom right sub matrix).
- If the bottom sub diagonal element switches between two convergent numbers other than 0; then two eigenvalues = sum of shifts +eigenvalues(bottom right sub matrix).
- Goto *
- If two eigenvalues are found, deflate matrix by 2 rows and 2 columns.
- If one eigenvalue is found, deflate matrix by 1 row and 1 column.
- Repeat process from * on deflated matrix
- Repeat until the matrix is deflated to 0 or 1x 1
- Output all the eigenvalues

. [B12]

(b)

$$A := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 9 & 8 & 7 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

QR(A)

$$\left[\begin{bmatrix} 0.1961161351 & 0.08514725113 & 0.1399183859 & 0.9668046609 \\ 0.9805806756 & -0.01702945022 & -0.02798367719 & -0.1933609321 \\ 0 & 0.9962228382 & -0.01243718986 & -0.08593819208 \\ 0 & 0 & 0.9896893832 & -0.1432303201 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 5.099019513 & 6.275716324 & 7.452413135 & 8.629109946 \\ -1.541101964 \cdot 10^{-13} & 9.034123345 & 8.106018307 & 7.17791327 \\ 2.119983221 \cdot 10^{-13} & 1.695986577 \cdot 10^{-13} & 6.062508198 & 5.197190713 \\ 7.238056756 \cdot 10^{-13} & 6.808185808 \cdot 10^{-14} & -2.385456482 \cdot 10^{-12} & 1.00261224 \end{bmatrix} \right]$$

[A2]

(c)

$$\begin{bmatrix} 5.099019513 & 6.275716324 & 7.452413135 & 8.629109946 \\ -1.541101964 \cdot 10^{-13} & 9.034123345 & 8.106018307 & 7.17791327 \\ 2.119983221 \cdot 10^{-13} & 1.695986577 \cdot 10^{-13} & 6.062508198 & 5.197190713 \\ 7.238056756 \cdot 10^{-13} & 6.808185808 \cdot 10^{-14} & -2.385456482 \cdot 10^{-12} & 1.00261224 \end{bmatrix}$$

$$
\cdot \begin{bmatrix} 0.1961161351 & 0.08514725113 & 0.1399183859 & 0.9668046609 \\ 0.9805806756 & -0.01702945022 & -0.02798367719 & -0.1933609321 \\ 0 & 0.9962228382 & -0.01243718986 & -0.08593819208 \\ 0 & 0 & 0.9896893832 & -0.1432303201 \end{bmatrix} \Bigg]
$$

$$
\begin{bmatrix} 7.153846153 & 7.751559661 & 8.985280383 & 1.839880381 \\ 8.858686773 & 7.92155441 & 6.750280476 & -3.471557884 \\ 2.078814579 \cdot 10^{-13} & 6.039609123 & 5.068203905 & -1.265396283 \\ 2.08709726 \cdot 10^{-13} & -2.315975559 \cdot 10^{-12} & 0.9922746893 & -0.143604472 \end{bmatrix}
$$

[M1,A1]

(d)

```
(ITERATES(RQ(M), M, A, 20))
```

[19, 20, 21]

$$
\Bigg[\begin{bmatrix} 19.2279091 & 4.590653468 & -5.035422208 & 1.808175985 \\ -1.4185317 \cdot 10^{-12} & 0.3420337057 & -4.085367446 & -2.383523882 \\ -1.235900053 \cdot 10^{-10} & 2.357353189 & 1.848948532 & -2.472060278 \\ 1.149072249 \cdot 10^{-10} & 7.38005669 \cdot 10^{-11} & 1.927740415 \cdot 10^{-6} & -1.41889101 \end{bmatrix}
$$

$$
, \begin{bmatrix} 19.2279091 & -4.324076249 & -5.266111673 & -1.808178341 \\ -3.63365835 \cdot 10^{-11} & 1.572327187 & -2.178848373 & 2.788689625 \\ -6.43274248 \cdot 10^{-11} & 4.26387351 & 0.6186559474 & -2.003863568 \\ 4.586951506 \cdot 10^{-11} & -3.108113003 \cdot 10^{-11} & 6.34843238 \cdot 10^{-7} & -1.418891907 \end{bmatrix}
$$

$$
, \begin{bmatrix} 19.2279091 & -6.436934524 & 2.235047921 \\ -1.667505682 \cdot 10^{-11} & 1.409642583 & -4.323864624 \\ 9.507009599 \cdot 10^{-11} & 2.118857516 & 0.7813396209 \\ 3.470346188 \cdot 10^{-10} & -1.478936 \cdot 10^{-10} & 3.98919629 \cdot 10^{-7} \end{bmatrix}
$$

$$\begin{bmatrix} 1.80817897 \\ 0.9152706603 \\ 3.30976442 \\ -1.418890976 \end{bmatrix}$$

-1.418890976

[M2,A2]

(e)

the matrix has settled to a quasi upper triangular matrix, so we can extract the eigenvalues

$$\begin{bmatrix} 19.2279091 & -6.436934524 & 2.235047921 & 1.80817897 \\ -1.667505682\cdot10^{-11} & 1.409642583 & -4.323864624 & 0.9152706603 \\ 9.507009599\cdot10^{-11} & 2.118857516 & 0.7813396209 & 3.30976442 \\ 3.470346188\cdot10^{-10} & -1.478936\cdot10^{-10} & 3.98919629\cdot10^{-7} & -1.418890976 \end{bmatrix}$$

ROW [2, 3] COL [2, 3]

$$\begin{bmatrix} 1.409642582 & -4.323864624 \\ 2.118857516 & 0.7813396208 \end{bmatrix}$$

$$\text{EIGENVALUES} \begin{bmatrix} 1.409642582 & -4.323864624 \\ 2.118857516 & 0.7813396208 \end{bmatrix}$$

[1.095491101 + 3.010475361·î, 1.095491101 − 3.010475361·î]

19.2279091

eigenvalues are

[19.2279091, 1.095491101 + 3.010475361·î, 1.095491101 − 3.010475361·î,
-1.418890976]

[M2,A3]